

Polynomials & Complex Numbers

If a polynomial's coefficients are all real then the roots will appear in complex conjugate pairs.

Every polynomial of degree n can be;

- factorised as a mixture of quadratic and linear factors over the real field
- factorised to n linear factors over the complex field

NOTE: odd ordered polynomials must have a real root

$$\begin{aligned} \text{e.g. (i) } x^2 + 2x + 2 &= (x + 1)^2 + 1 \\ &= \underline{(x + 1 + i)(x + 1 - i)} \end{aligned}$$

$$(ii) z^4 + z^2 - 12 = 0$$

$$(z^2 - 3)(z^2 + 4) = 0$$

$$(z + \sqrt{3})(z - \sqrt{3})(z^2 + 4) = 0 \quad (\text{factorised over Real numbers})$$

$$(z + \sqrt{3})(z - \sqrt{3})(z + 2i)(z - 2i) = 0 \quad (\text{factorised over Complex numbers})$$

$$\underline{z = \pm\sqrt{3} \quad \text{or} \quad z = \pm 2i}$$

If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$

If $(ax - b)$ is a factor of $P(x)$, then $P\left(\frac{b}{a}\right) = 0$

$$(iii) \text{ Factorise } 2x^3 - 3x^2 + 8x + 5$$

as it is a cubic it must have a real factor

$$\begin{aligned} P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 8\left(-\frac{1}{2}\right) + 5 & \therefore 2x^3 - 3x^2 + 8x + 5 \\ &= -\frac{1}{4} - \frac{3}{4} - 4 + 5 & = (2x + 1)(x^2 - 2x + 5) \\ &= 0 & = (2x + 1)[(x - 1)^2 + 4] \\ & & = \underline{(2x + 1)(x - 1 - 2i)(x - 1 + 2i)} \end{aligned}$$

(iv) Given that $P(x) = 4x^4 + 8x^3 + 5x^2 + x - 3$ has two rational zeros, find these zeros and factorise $P(x)$ over the complex field.

$$\begin{aligned} P\left(\frac{1}{2}\right) &= 4\left(\frac{1}{16}\right) + 8\left(\frac{1}{8}\right) + 5\left(\frac{1}{4}\right) + \frac{1}{2} - 3 \\ &= 0 \end{aligned}$$

$\therefore (2x - 1)$ is a factor

$$P(x) = (2x - 1)(2x^3 + 5x^2 + 5x + 3)$$

$$\begin{aligned} P\left(-\frac{3}{2}\right) &= 2\left(\frac{-27}{8}\right) + 5\left(\frac{9}{4}\right) + 5\left(-\frac{3}{2}\right) + 3 \\ &= 0 \end{aligned}$$

$\therefore (2x + 3)$ is a factor

\therefore rational zeros are $\frac{1}{2}$ and $-\frac{3}{2}$

$$P(x) = 4x^4 + 8x^3 + 5x^2 + x - 3$$

$$= (2x - 1)(2x + 3)(x^2 + x + 1)$$

$$= (2x - 1)(2x + 3) \left[\left(x + \frac{1}{2} \right)^2 + \frac{3}{4} \right]$$

$$= (2x - 1)(2x + 3) \left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i \right)$$

$$1 \times 3 \times 2x = 6x$$

$$1 \times 2x \times -1 = -2x$$

$$4x$$

$$-1 \times 3 \times ?x = -3x$$

(v) Solve $z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$ given that it has a real root of multiplicity 2

real roots must be a divisor of 45 i.e. 1, 3, 5, 9, 15, 45

$$P'(z) = 4z^3 - 18z^2 + 28z - 30$$

$$P'(1) \neq 0 \qquad P'(3) = 4(3)^3 - 18(3)^2 + 28(3) - 30 = 0$$

$$P(3) = (3)^4 - 6(3)^3 + 14(3)^2 - 30(3) + 45 = 0$$

$\therefore (z - 3)^2$ is a factor

$$z^4 - 6z^3 + 14z^2 - 30z + 45 = 0$$

$$(z - 3)^2(z^2 + 5) = 0$$

$$(z - 3)^2(z - \sqrt{5}i)(z + \sqrt{5}i) = 0$$

$$\underline{z = 3 \text{ or } z = \pm\sqrt{5}i}$$

Exercise 1G; 1b, 2, 3, 5, 6, 7b, 8 to 19, 22a to c

Patel: Exercise 5B, 6b, 7b, 8acegh