

Differentiating Inverse Trig

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{\cos^2 y}}$$

$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

$$y = \cos^{-1} x$$

$$x = \cos y$$

$$\frac{dx}{dy} = -\sin y$$

$$\frac{dy}{dx} = \frac{-1}{\sin y}$$

$$= \frac{-1}{\sqrt{\sin^2 y}}$$

$$= \frac{-1}{\sqrt{1 - \cos^2 y}}$$

$$= \frac{-1}{\sqrt{1 - x^2}}$$

$$y = \tan^{-1} x$$

$$x = \tan y$$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$= \frac{1}{1 + \tan^2 y}$$

$$= \frac{1}{1 + x^2}$$

In general;

$$y = \sin^{-1} \frac{x}{a}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$y = \cos^{-1} \frac{x}{a}$$
$$\frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$y = \tan^{-1} \frac{x}{a}$$
$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$y = \sin^{-1} f(x)$$
$$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \cos^{-1} f(x)$$
$$\frac{dy}{dx} = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \tan^{-1} f(x)$$
$$\frac{dy}{dx} = \frac{f'(x)}{1 + [f(x)]^2}$$

e.g. (i) $y = \sin^{-1} 5x$

$$\frac{dy}{dx} = \frac{5}{\sqrt{1 - 25x^2}}$$

(ii) $y = \cos^{-1} e^x$

$$\frac{dy}{dx} = \frac{-e^x}{\sqrt{1 - e^{2x}}}$$

$$(iii) y = \sin^{-1}\left(\frac{x}{3}\right)$$

OR

$$y = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{1}{3}}{\sqrt{1 - \frac{x^2}{9}}} \\ &= \frac{\frac{1}{3}}{\frac{1}{3}\sqrt{9 - x^2}} \\ &= \frac{1}{\sqrt{9 - x^2}} \end{aligned}$$

$$\underline{\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}}$$

$$(iv) y = e^{\cos^{-1} x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{\sqrt{1-x^2}} e^{\cos^{-1} x} \\ &= \frac{-e^{\cos^{-1} x}}{\sqrt{1-x^2}} \end{aligned}$$

$$(v) y = (\tan^{-1} x)^3$$

$$\begin{aligned} \frac{dy}{dx} &= 3(\tan^{-1} x)^2 \cdot \frac{1}{1+x^2} \\ &= \frac{3(\tan^{-1} x)^2}{1+x^2} \end{aligned}$$

$$(vi) y = x^2 \tan^{-1} x^3$$

$$\begin{aligned} \frac{dy}{dx} &= (x^2) \left(\frac{3x^2}{1+x^6} \right) + (\tan^{-1} x^3)(2x) \\ &= \frac{3x^4}{1+x^6} + 2x \tan^{-1} x^3 \end{aligned}$$

(vii) **2018 HSC Question 12c)** Let $f(x) = \sin^{-1} x + \cos^{-1} x$

a) Show that $f'(x) = 0$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$$
$$= 0$$

b) Hence, or otherwise, prove $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

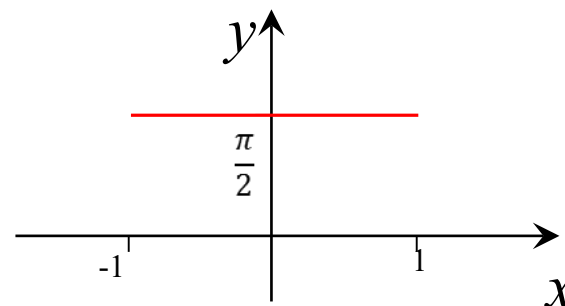
Since $f'(x) = 0$ and both $\sin^{-1} x$ and $\cos^{-1} x$ are continuous functions over their domain, then $f(x)$ is a constant

$$\text{When } x = 0; f(x) = \sin^{-1} 0 + \cos^{-1} 0$$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

c) Hence, sketch $f(x) = \sin^{-1} x + \cos^{-1} x$



(viii) Let $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$

a) Show that $f'(x) = 0$

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{\frac{-1}{x^2}}{1+\frac{1}{x^2}} \\ &= \frac{1}{1+x^2} + \frac{-1}{x^2+1} = \underline{0} \end{aligned}$$

b) Sketch $y = f(x)$

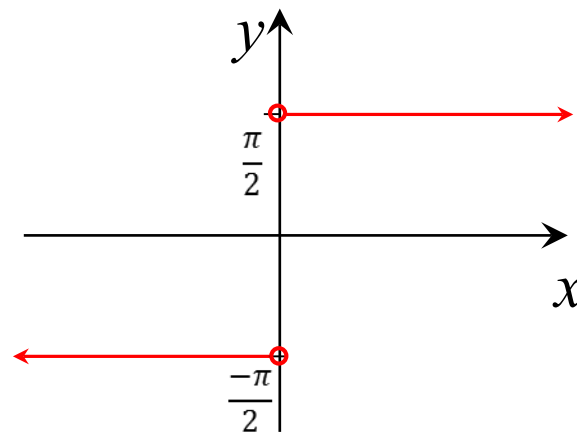
$\tan^{-1} \frac{1}{x}$ is discontinuous at $x = 0$

When $x = 1$; $f(x) = \tan^{-1} 1 + \tan^{-1} 1$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

When $x = -1$; $f(x) = \tan^{-1} -1 + \tan^{-1} -1$

$$= \frac{-\pi}{4} + \frac{-\pi}{4} = -\frac{\pi}{2}$$



Exercise 12A; 2ace etc, 3bd, 4a, 5, 6a, 9ace etc, 11, 13, 15, 16, 18, 20