Integration By Parts

DIFF	INT	+ /–
u	dv	
$\frac{du}{dx}$	v	+
	$\int v dx$	–

$$\int u dv = uv - \int v du$$

u should be chosen so that differentiation makes it a simpler function.

dv should be chosen so that it can be integrated

Case 1: polynomial times integratable function

Differentiate the polynomial down to zero

e.g. (i) $\int x\cos x\,dx$

$= x\sin x \; - -\cos x$

$= x\sin x + \cos x + c$

DIFF	INT	+ /–
x	$\cos x$	
1	$\sin x$	+
0	$-\cos x$	–

(ii) $\int x^2 e^x\,dx$

$= x^2 e^x \; -2xe^x + 2e^x + c$

DIFF	INT	+ /–
x^2	e^x	
$2x$	e^x	+
2	e^x	–
0	e^x	+

Case 2: polynomial times non-integratable function

Stop when product of the line is integratable

$$(i)\int \log x\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + c$$

DIFF	INT	+ /–
$\ln x$	1	
$\frac{1}{x}$	x	$+$
		$-$

$$(ii)\int \tan^{-1} x\,dx$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

DIFF	INT	+ /–
$\tan^{-1} x$	1	
$\frac{1}{1+x^2}$	x	$+$
		$-$

$$(iii)\int x^3\log x\,dx$$

$$=\frac{1}{4}x^4\log x-\frac{1}{4}\int x^3\,dx$$

$$=\frac{1}{4}x^4\log x-\frac{1}{16}x^4+c$$

DIFF	INT	+/–
$\log x$	x^3	
$\frac{1}{x}$	$\frac{1}{4}x^4$	+
		–

OR

$$\int x^3\log x\,dx$$

$$=x^4\log x-x^4-3\int\left(x^3\log x-x^3\right)dx$$

$$\therefore 4\int x^3\log x\,dx=x^4\log x-x^4+3\int x^3\,dx$$

$$=x^4\log x-x^4+\frac{3}{4}x^4+c$$

$$=x^4\log x-\frac{1}{4}x^4+c$$

$$\int x^3\log x\,dx=\frac{1}{4}x^4\log x-\frac{1}{16}x^4+c$$

DIFF	INT	+/–
x^3	$\log x$	
$3x^2$	$x\log x-x$	+
		–

Case 3: two integratable functions

Stop when product of the line is integratable or a multiple of another line

$$(i)\int e^{x}\cos x\,dx$$

$$= e^{x}\sin x - e^{x}\cos x - \int e^{x}\cos x\,dx$$

$$\therefore 2\int e^{x}\cos x\,dx = e^{x}\sin x - e^{x}\cos x + c$$

$$\int e^{x}\cos x\,dx = \frac{1}{2}e^{x}\sin x - \frac{1}{2}e^{x}\cos x + c$$

DIFF	INT	+ /–
e^{x}	$\cos x$	
e^{x}	$\sin x$	+
e^{x}	$-\cos x$	–
		+

Other examples

$$\int xe^{xdx}$$

$$= xe^{x} - e^{x} + c$$

DIFF	INT	+ /–
x	e^{x}	
1	e^{x}	+
0	e^{x}	–

$$\int \sin^{-1} x\,dx$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^{2}}}dx$$

$$= x\sin^{-1} x + \sqrt{1-x^{2}} + c$$

DIFF	INT	+ /–
$\sin^{-1} x$	1	
$\frac{1}{\sqrt{1-x^{2}}}$	x	+
		–

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx$$

$$= -\frac{\sqrt{1-x^2}}{x} - \int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x + c$$

DIFF	INT	+ /–
$\sqrt{1-x^2}$	$\frac{1}{x^2}$	
$-\frac{x}{\sqrt{1-x^2}}$	$-\frac{1}{x}$	+
		–

Using Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\int e^x \cos x\,dx = Re\left(\int e^x(\cos x + i\sin x)dx\right)$$

$$= Re\left(\int e^x e^{ix}\,dx\right)$$

$$= Re\left(\int e^{(1+i)x}\,dx\right)$$

$$= Re\left(\frac{1}{1+i}e^{(1+i)x}\right) + c$$

$$= \frac{1}{2}Re\left((1-i)e^{(1+i)x}\right) + c$$

$$= \frac{1}{2}e^x\,Re((1-i)(\cos x + i\sin x)) + c$$

$$= \frac{1}{2}e^x(\cos x + \sin x) + c$$