Investing Money by Regular Instalments

Annuities

An annuity is a compound interest investment from which payments are made or received on a regular basis for a fixed period of time.

ordinary annuity: deposit the same amount at the end of each compounding period e.g. loans

annuity due: deposit the same amount at the beginning of each compounding period e.g. superannuation fund

future value (FV): the total value of the investment at the end of the term of the investment.

present value (PV): the single amount that could be initially invested to produce the same FV over a given time period

Note: annuities are not to be confused with the financial product known as an "annuity", whilst this product is a type of annuity, it is NOT the only type.

Annuity Due

2002 HSC Question 9b)

A superannuation fund pays an interest rate of 8.75% p.a. which compounds annually. Stephanie decides to invest $5000 in the fund at the beginning of each year, commencing on 1 January 2003.

a) What will be the value of Stephanie's superannuation when she retires on 31 December 2023?

$A_{21} = 5000(1.0875)^{21}$ amount invested for 21 years

$A_{20} = 5000(1.0875)^{20}$ amount invested for 20 years

$A_{19} = 5000(1.0875)^{19}$ amount invested for 19 years

$\vdots$

$\vdots$

$A_{1} = 5000(1.0875)^{1}$ amount invested for 1 year

$$\text{Amount} = 5000(1.0875)^{21} + 5000(1.0875)^{20} \ldots + 5000(1.0875)$$

$$a = 5000(1.0875), r = 1.0875, n = 21$$

$$= S_{21}$$

$$= \frac{5000(1.0875)(1.0875^{21} - 1)}{0.0875}$$

$$= \$299604.86$$

b*) Find the year when the fund first exceeds \$200000.

$$\text{Amount} = 5000(1.0875) + 5000(1.0875)^2 + \ldots + 5000(1.0875)^n$$

$$= S_n$$

i.e $S_n > 200000$

$$\frac{5000(1.0875)\left(1.0875^n - 1\right)}{0.0875} > 200000$$

$$\left(1.0875^n - 1\right) > \frac{280}{87}$$

$$1.0875^n > \frac{367}{87}$$

$$\log\left(1.0875^n\right) > \log\left(\frac{367}{87}\right)$$

$$n\log(1.0875) > \log\left(\frac{367}{87}\right)$$

$$n > \frac{\log\left(\frac{367}{87}\right)}{\log(1.0875)}$$

$$n > 17.16056585$$

$$\therefore n = 18$$

Thus 2021 is the first year when the fund exceeds $200000

c*) What annual instalment would have produced \$1000000 by 31st December 2020?

$$\text{Amount} = P(1.0875)^{18} + P(1.0875)^{17} + \ldots + P(1.0875)$$

$$a = P(1.0875), r = 1.0875, n = 18$$

i.e. $S_{18} = 1000000$

$$\frac{P(1.0875)\left(1.0875^{18} - 1\right)}{0.0875} = 1000000$$

$$P = \frac{(1000000)(0.0875)}{1.0875\left(1.0875^{18} - 1\right)}$$

$$= 22818.16829$$

An annual instalment of \$22818.17 will produce \$1000000

Future Value of Annuity Due

$$FV = PR + PR^2 + PR^3 + \ldots PR^n$$

$$= \frac{PR(R^n - 1)}{R - 1}$$

$P =$ principal

$R = 1 +$ interest rate as a decimal(or fraction)

$n =$ time periods

d*) If Stephanie could afford it, how much money could she save by making one investment and still have the same future value in 21 years time?

$$PV(1.0875)^{21} = 299604.86$$

$$PV = \frac{299604.86}{(1.0875)^{21}}$$

$$= 54467.64$$

A present value of $54 467.64 would need to be invested

$$\text{actual investment} = \$5000 \times 21$$
$$= \$105\ 000$$

$$\text{saving} = 105000 - 54467.64$$
$$= 50532.36$$

Stephanie would save \$50 532.36

Present Value of Annuity Due

$$(PV)R^n = FV$$
$$= \frac{PR(R^n - 1)}{R - 1}$$
$$PV = \frac{PR\left(1 - R^{-n}\right)}{R - 1}$$

$P =$ principal

$R = 1 +$ interest rate as a decimal(or fraction)

$n =$ time periods

(ii) A table of future value interest factors is used to compare annuities of $1

Table of future value interest factors						
Period	Interest rate per period					
	0.50%	1.50%	2.00%	3.00%	4.00%	6.00%
4	4.0301	4.0909	4.1216	4.1836	4.2465	4.3746
8	8.1414	8.4328	8.5830	8.8923	9.2142	9.8975
16	16.6142	17.9324	18.6393	20.1569	21.8245	25.6725
32	34.6086	40.6883	44.2270	52.5028	62.7015	90.8898
48	54.0978	69.5652	79.3535	104.4084	139.2632	256.5645
96	122.8285	211.7202	284.6467	535.8502	1054.2960	4462.6505

Tim and Janene were offered interest of 6% p.a. for 4 years.

Tim invested $1500 each quarter compounded quarterly and Janene invested $500 every month compounded monthly.

Which person earned the greater amount of interest?

Tim: 6% p.a = 1.5% per quarter

$FV = 17.9324 \times 1500$

$= 26898.6$

Janene: 6% p.a = 0.5% per month

$FV = 54.0978 \times 500$

$= 27048.9$

$$\text{Tim's interest} = 26\,898.60 - 1500 \times 16$$
$$= 2\,898.60$$

$$\text{Janene's interest} = 27\,048.90 - 500 \times 48$$
$$= 3\,048.90$$

Janene earned \$150.30 more interest than Tim

Exercise 14C; 5, 9, 12, 13, 17