Standardising a Normal Distribution

Let $Z \sim N(0,1)$

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

1. $y = \varphi\left(\frac{z}{\sigma}\right)$ will stretch the graph horizontally by a factor of σ

$$\varphi\left(\frac{z}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$$

However, stretching horizontally will also increase the area under the curve by a factor of σ, and the area under the pdf must always equal one

$bh = 1$ Area $= 1 \Rightarrow$ $(\sigma b)h = 1$ Area $= \sigma \Rightarrow$ $(\sigma b)(\frac{h}{\sigma}) = 1$ Area $= 1$

To correct this the graph needs to be stretched vertically by a factor of $\frac{1}{\sigma}$

2. $\dfrac{y}{\frac{1}{\sigma}} = \varphi\left(\dfrac{z}{\sigma}\right)$ will stretch vertically by a factor of $\dfrac{1}{\sigma}$

$$\frac{1}{\sigma}\varphi\left(\frac{z}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{z^2}{2\sigma^2}}$$

3. $y = \dfrac{1}{\sigma}\varphi\left(\dfrac{z-\mu}{\sigma}\right)$ will shift to the right μ units

$$\frac{1}{\sigma}\varphi\left(\frac{z-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(z-\mu)^2}{2\sigma^2}}$$

$\therefore Z \sim N(0,1) \iff X \sim N(\mu,\sigma^2)$

where $X = \dfrac{Z-\mu}{\sigma}$

z - scores

All normal distributions can be converted to a standard normal distribution using the **standardisation equation**

$$Z = \frac{X - \mu}{\sigma}$$

The result of the conversion is known as the score's ***z*-score**

A *z*-score represents the number of standard deviations a score is above/below the mean, and is usually given to two decimal places.

Quartiles & Outliers

$$\Phi(Q_3) = 0.75$$

$$Q_3 = 0.6745 = 0.67$$

$$\text{by symmetry } Q_1 = -0.67$$

$$IQR = Q_3 - Q_1$$

$$= 1.349 = 1.35$$

$$\text{outlier} > Q_3 + 1.5 \times IQR$$

$$= 0.6745 + 1.5 \times 1.349$$

$$= 2.698 = 2.70$$

∴ an outlier is any score where; $|z\text{-score}| > 2.70$

e.g. (i) The time spent waiting for a prescription to be prepared at a pharmacy is normally distributed with mean 15 minutes and standard deviation 2.8 minutes.

Find the probability that the waiting time is more than 20 minutes.

Let W = waiting time in minutes

$W \sim N(15, 7.84)$

$$P(W > 20) = P\left(Z > \frac{20 - 15}{2.8}\right)$$

$$= 1 - \Phi\left(\frac{25}{14}\right)$$

$$= 1 - 0.96293$$

$$= 0.03707$$

(ii) The T-Q company makes a soft drink sold in 330 mL cans. The actual volume of drink in the cans is distributed normally with standard deviation 2.5 mL.

To ensure that at least 99% of the cans contain more than 330 mL, find the volume that the company should supply in the cans on average.

Let V = volume of drink

$V \sim N(\mu, 6.25)$

$$P(V > 330) = P\left(Z > \frac{330 - \mu}{2.5}\right)$$

$$\Phi\left(\frac{330 - \mu}{2.5}\right) = 0.01$$

$$\frac{330 - \mu}{2.5} = -2.33$$

$$330 - \mu = -5.825$$

$$\mu = 335.825$$

The company should supply 336 mL in the cans

(iii) A biologist has been collecting data on the heights of a particular species of cactus. They have observed that 34.2% of the cacti are below 12 cm in height and 18.4% of the cacti are above 16 cm in height.

Assuming that the heights are normally distributed. Find the mean and standard deviation of the distribution.

Let $H =$ the height of cactus $\quad H \sim N(\mu, \sigma^2)$

$$P\left(Z < \frac{12 - \mu}{\sigma}\right) = 0.342 \qquad P\left(Z > \frac{16 - \mu}{\sigma}\right) = 0.184$$

$$\Phi\left(\frac{12 - \mu}{\sigma}\right) = 0.342 \qquad \Phi\left(\frac{16 - \mu}{\sigma}\right) = 0.816$$

$$\frac{12 - \mu}{\sigma} = -0.41 \qquad \frac{16 - \mu}{\sigma} = 0.90$$

$$12 - \mu = -0.41\sigma \qquad 16 - \mu = 0.90\sigma$$

$$12 - \mu = -0.41\sigma$$
$$16 - \mu = 0.90\sigma$$

$$4 = 1.31\sigma$$
$$\sigma = 3.05 \qquad \therefore \mu = 13.25$$

Exercise 16E; 1ace, 3adf, 4ace, 6acd, 9, 10, 11

Exercise 16F; 1, 2, 4, 5, 6, 7, 9, 11