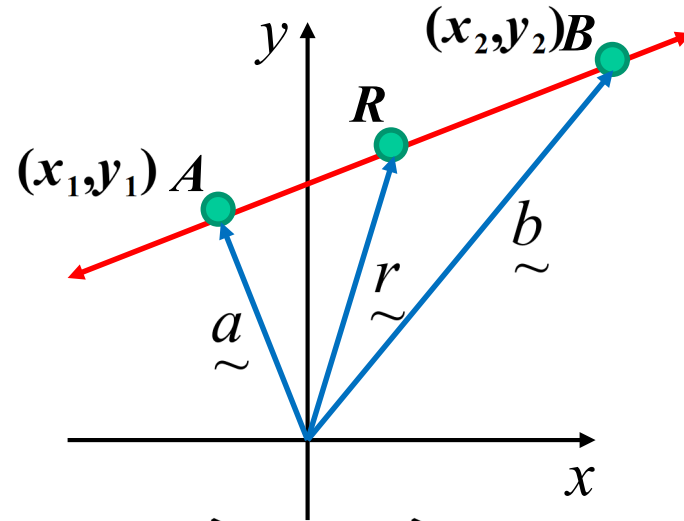


Vector Equation of a Line in 2D



If R is on the line AB then $\vec{AR} = \lambda \vec{AB}$, $\lambda \in \mathbb{R}$

$$\vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$$

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \quad \text{vector equation of a line}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$x = x_1 + \lambda(x_2 - x_1)$$

parametric equation of a line

$$y = y_1 + \lambda(y_2 - y_1)$$

$$\therefore \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \lambda$$

cartesian equation of a line

$$(y_2 - y_1)(x - x_1) = (x_2 - x_1)(y - y_1)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - y_1 = m(x - x_1)$$

e.g. Write a vector equation of the line passing through $(3, -5)$ and $(-2, -8)$

$$\underline{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \end{pmatrix}}$$

NOTE
cartesian equation

$$y = \frac{3}{5}x - \frac{34}{3}$$

OR $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

notice the similarity
to the slope

Recall from complex numbers, rotation of a vector 90° is multiplication by i

$$\begin{aligned}\begin{pmatrix} \textit{run} \\ \textit{rise} \end{pmatrix} \times i &= (\textit{run} + i\textit{rise})(i) \\ &= (i\textit{run} - \textit{rise}) \\ &= \begin{pmatrix} -\textit{rise} \\ \textit{run} \end{pmatrix}\end{aligned}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} -b \\ a \end{pmatrix}$$

and using the dot product

$$\begin{aligned}\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} &= -ab + ab \\ &= 0\end{aligned}$$

$$\therefore \begin{pmatrix} a \\ b \end{pmatrix} \perp \begin{pmatrix} -b \\ a \end{pmatrix}$$

(ii) Find a vector equation for the line $2x + 5y - 1 = 0$

$$\underline{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

any point that
lies on the line

looks like $\frac{1}{m}$

(iii) Find the point of intersection of $2x + y + 1 = 0$ and $3x + 5y - 9 = 0$

$$\underline{r} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} \lambda &= 3 + 5\mu \\ -1 - 2\lambda &= -3\mu \end{aligned}$$

$$\underline{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

$$\begin{aligned} -1 - 2(3 + 5\mu) &= -3\mu \\ 7\mu &= -7 \\ \mu &= -1 \end{aligned}$$

$$\therefore \underline{r} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

point of intersection is $(-2, 3)$

(iv) Find a vector equation for the line joining the points $(-1,1)$ and $(4,11)$.
Use this to write parametric equations for any point on the line .
Hence find the coordinates of the points where the line meets the
parabola $y = x^2$

$$\vec{r} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x = -1 + \lambda$$

$$y = 1 + 2\lambda$$

$$y = x^2$$

$$1 + 2\lambda = (-1 + \lambda)^2$$

$$1 + 2\lambda = 1 - 2\lambda + \lambda^2$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0 \quad \text{or} \quad \lambda = 4$$

\therefore parabola meets the line at $(-1,1)$ and $(3,9)$

Vector Equation of a Line in 3D

The equation of the line that passes through A and B is given by;

$$\underline{\underline{\underline{r} = \underline{a} + \lambda \underline{b}}}}$$

where: \underline{a} is any point on AB

$$\underline{b} = \overrightarrow{AB} \quad (\text{direction vector})$$

e.g. (i) find a vector equation of the line through $(1,2,3)$ and $(-2,7,4)$

$$\underline{\underline{\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}}}}$$

(ii) What is its corresponding Cartesian equation?

$$\underline{\underline{\underline{\frac{x-1}{3} = \frac{2-y}{5} = 3-z}}}}$$

(iii) Find the vector equation of the line that passes through $(-2, 1, 4)$ and is parallel to $2\vec{i} + \vec{j} - 2\vec{k}$

$$\underline{\vec{r} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}}$$

two lines are parallel if their direction vectors are scalar multiples

(iv) Find the vector equation of a line that passes through $(0, 2, 3)$ and is perpendicular to $\vec{i} - \vec{j} + 4\vec{k}$

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$a - b + 4c = 0$$

$$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

two lines are perpendicular if the dot product of their direction vectors equal zero

Also the two lines must intersect

$$\lambda a = \mu$$

$$2 + \lambda b = -\mu$$

$$3 + \lambda c = 4\mu$$

$$2 + \lambda b = -\lambda a \Rightarrow \lambda = -\frac{2}{a+b}$$

$$3 + \lambda c = 4\lambda a \Rightarrow \lambda = \frac{3}{4a-c}$$

$$\frac{3}{4a-c} = -\frac{2}{a+b}$$

$$3a + 3b = -8a + 2c$$

$$11a + 3b - 2c = 0$$

$$11a - 11b + 44c = 0$$

$$14b - 46c = 0$$

$$b = \frac{23c}{7}$$

let $c = 7$

$\therefore b = 23$ and $a = -5$

$$\underline{\tilde{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 23 \\ 7 \end{pmatrix}}$$

if two lines have direction vectors \underline{b}_1 and \underline{b}_2 they are;

parallel if $\underline{b}_1 = \mu \underline{b}_2$, $\mu \in \mathbb{R}$

perpendicular if $\underline{b}_1 \cdot \underline{b}_2 = 0$ and the lines intersect

skew if the lines are not parallel and do not intersect

(v) Show that lines $\underline{r}_1 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\underline{r}_2 = \mu \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ are skewed

$$1 + \lambda = \mu \dots \textcircled{1}$$

$$3 + \lambda = 4\mu \dots \textcircled{2}$$

$$-1 = 5\mu \dots \textcircled{3}$$

equate $\textcircled{1}$ and $\textcircled{3}$

equate $\textcircled{2}$ and $\textcircled{3}$

$$\mu = -\frac{1}{5} \quad 1 + \lambda = -\frac{1}{5} \quad 3 + \lambda = -\frac{4}{5}$$

$$\lambda = -\frac{6}{5} \quad \lambda = -\frac{19}{5} \neq -\frac{6}{5}$$

need to show that there is no point of intersection

thus the lines do not intersect and are not parallel
 \therefore the lines are skewed

**Exercise 5F; 1, 2, 3a, 4b, 5ac, 7a, 9a, 11a, 12b,
14, 15b, 17, 18, 20, 21, 24, 25, 26**