Rates of Change & Exponentials

A common application involving the exponential function is the growth & decay of objects.

It starts with the assumption that growth and decay is proportional to population.

$$\frac{dP}{dt} = kP$$

This simple model can be applied to things such as population growth and the decay of radioactive material

The solution of this differential equation is;

$$P = Ae^{kt}$$

e.g (2015 HSC)

The amount of caffeine, C, in the human body decreases according to the equation $\frac{dC}{dt} = -0.14C$, where C is measured in mg and t is the time in hours.

(i) Show that $C = Ae^{-0.14t}$ is a solution to $\frac{dC}{dt} = -0.14C$, where A is a constant

$$C = Ae^{-0.14t}$$

$$\frac{dC}{dt} = -0.14Ae^{-0.14t}$$

$$\therefore \frac{dC}{dt} = -0.14C$$

When $t = 0$, there are 130 mg of caffeine in Lee's body.

(*ii*) Find the value of A.

When $t = 0$, $C = 130$

$$\therefore 130 = Ae^{0}$$
$$A = 130$$

(*iii*) What is the amount of caffeine in Lee's body after 7 hours?

When $t = 7$, $C = 130e^{-0.14(7)}$

$$= 48.7904...$$
$$= 48.8 \quad \text{(to 1 dp)}$$

$\therefore$ 48.8 mg of caffeine remains in Lee's body

(iv) What is the time taken for the amount of caffeine in Lee's body to halve?

When $C = 65$,
$$65 = 130e^{-0.14t}$$
$$e^{-0.14t} = \frac{1}{2}$$
$$-0.14t = \ln\frac{1}{2}$$
$$t = -\frac{1}{0.14}\ln\frac{1}{2}$$
$$t = 4.95105\ldots$$

$\therefore$ it takes about 5 hours for the amount of caffeine to halve

Exercise 11F; 1, 3, 5, 7, 8, 9, 11