

Relationships Between Binomial Coefficients

Binomial Theorem $(1+x)^n = \sum_{k=0}^n {}^n C_k x^k$
 $= {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_k x^k + \dots + {}^n C_n x^n$

e.g. (i) Find the values of;

a) $\sum_{k=1}^n {}^n C_k$

$$(1+x)^n = \sum_{k=0}^n {}^n C_k x^k$$

$$\sum_{k=1}^n {}^n C_k = 2^n - {}^n C_0$$

let $x = 1$; $(1+1)^n = \sum_{k=0}^n {}^n C_k 1^k$

$$\sum_{k=1}^n {}^n C_k = 2^n - 1$$

$$2^n = \sum_{k=0}^n {}^n C_k$$

$$2^n = {}^n C_0 + \sum_{k=1}^n {}^n C_k$$

$$\text{b) } {}^n C_1 + {}^n C_3 + {}^n C_5 + {}^n C_7 + \dots$$

$$\begin{aligned} (1+x)^n &= \sum_{k=0}^n {}^n C_k x^k \\ &= {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + {}^n C_3 x^3 + {}^n C_4 x^4 + {}^n C_5 x^5 + \dots \end{aligned}$$

$$\text{let } x = 1; (1+1)^n = {}^n C_0 + {}^n C_1 + {}^n C_2 + {}^n C_3 + {}^n C_4 + {}^n C_5 + \dots$$

$$2^n = {}^n C_0 + {}^n C_1 + {}^n C_2 + {}^n C_3 + {}^n C_4 + {}^n C_5 + \dots \quad (1)$$

$$\text{let } x = -1; (1-1)^n = {}^n C_0 - {}^n C_1 + {}^n C_2 - {}^n C_3 + {}^n C_4 - {}^n C_5 + \dots$$

$$0 = {}^n C_0 - {}^n C_1 + {}^n C_2 - {}^n C_3 + {}^n C_4 - {}^n C_5 + \dots \quad (2)$$

subtract (2) from (1)

$$2^n = 2^n C_1 + 2^n C_3 + 2^n C_5 + \dots$$

$$\underline{2^{n-1} = {}^n C_1 + {}^n C_3 + {}^n C_5 + \dots}$$

$$\text{c) } \sum_{k=1}^n k^n C_k$$

$$(1+x)^n = \sum_{k=0}^n {}^n C_k x^k$$

Differentiate both sides

$$n(1+x)^{n-1} = \sum_{k=0}^n k^n C_k x^{k-1}$$

let $x = 1$;

$$n(1+1)^{n-1} = \sum_{k=0}^n k^n C_k$$

$$n(2)^{n-1} = (0)^n C_0 + \sum_{k=1}^n k^n C_k$$

$$\underline{\sum_{k=1}^n k^n C_k = n(2)^{n-1}}$$

(ii) By equating the coefficients of x^n on both sides of the identity;

show that; $(1+x)^n (1+x)^n \equiv (1+x)^{2n}$

$$\sum_{k=0}^n \binom{n}{k}^2 = \frac{(2n)!}{(n!)^2}$$

$$(1+x)^n = \sum_{k=0}^n {}^n C_k x^k$$

$$= {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_{n-2} x^{n-2} + {}^n C_{n-1} x^{n-1} + {}^n C_n x^n$$

coefficient of x^n in $(1+x)^n (1+x)^n$

$$\left({}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_{n-2} x^{n-2} + {}^n C_{n-1} x^{n-1} + {}^n C_n x^n \right)$$

$$\times \left({}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_{n-2} x^{n-2} + {}^n C_{n-1} x^{n-1} + {}^n C_n x^n \right)$$

$$= \binom{n}{0} \binom{n}{n} x^n + \binom{n}{1} x \binom{n}{n-1} x^{n-1} + \binom{n}{2} x^2 \binom{n}{n-2} x^{n-2}$$

$$+ \dots + \binom{n}{n-2} x^{n-2} \binom{n}{2} x^2 + \binom{n}{n-1} x^{n-1} \binom{n}{1} x + \binom{n}{n} x^n \binom{n}{0}$$

$$\text{coefficient of } x^n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \dots + \binom{n}{n}\binom{n}{0}$$

$$\text{But } \binom{n}{k} = \binom{n}{n-k}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2$$

$$= \sum_{k=0}^n \binom{n}{k}^2$$

coefficient of x^n in $(1+x)^{2n}$

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \dots + \binom{2n}{n}x^n + \dots + \binom{2n}{2n}x^{2n}$$

$$\text{coefficient of } x^n = \binom{2n}{n}$$

Now

$$(1+x)^n (1+x)^n \equiv (1+x)^{2n}$$

$$\begin{aligned} \therefore \sum_{k=0}^n \binom{n}{k}^2 &= \binom{2n}{n} \\ &= \frac{(2n)!}{n!n!} \\ &= \frac{(2n)!}{(n!)^2} \end{aligned}$$

Exercise 15D;
1, 2, 4, 5, 6, 7, 9, 11, 12