Mathematical Induction

$e.g.(iii)$ Prove $n(n+1)(n+2)$ is divisible by 3

Prove the result is true for $n = 1$

$$(1)(2)(3)$$

$= 6$ which is divisible by 3

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

i.e. $k(k + 1)(k + 2) = 3P$, where $P \in \mathbb{Z}$

Prove the result is true for $n = k + 1$

i.e. Prove $(k + 1)(k + 2)(k + 3) = 3Q$, where $Q \in \mathbb{Z}$

Proof:

$$(k+1)(k+2)(k+3)$$
$$= k(k+1)(k+2) + 3(k+1)(k+2)$$
$$= 3P + 3(k+1)(k+2) \qquad \text{(by assumption)}$$
$$= 3[P + (k+1)(k+2)]$$
$$= 3Q \ , \ \text{where } Q = P + (k+1)(k+2) \in \mathbb{Z}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

(iv) Prove $3^{3n} + 2^{n+2}$ is divisible by 5

Prove the result is true for $n = 1$

$$3^3 + 2^3$$

$$= 27 + 8$$

$= 35$ which is divisible by 5

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, $k \in \mathbb{Z}^+$

i.e. $3^{3k} + 2^{k+2} = 5P$, where $P \in \mathbb{Z}$

Prove the result is true for $n = k + 1$

i.e. Prove $3^{3k+3} + 2^{k+3} = 5Q$, where $Q \in \mathbb{Z}$

Proof: $3^{3k+3} + 2^{k+3}$

$$= 27 \cdot 3^{3k} + 2^{k+3}$$

$$= 27\left(5P - 2^{k+2}\right) + 2^{k+3}$$ ($3^{3k} = 5P - 2^{k+2}$ by assumption)

$$= 135P - 27 \cdot 2^{k+2} + 2^{k+3}$$

$$= 135P - 27 \cdot 2^{k+2} + 2 \cdot 2^{k+2}$$

$$= 135P - 25 \cdot 2^{k+2}$$

$$= 5\left(27P - 5 \cdot 2^{k+2}\right)$$

$= 5Q$, where $Q = 27P - 5 \cdot 2^{k+2} \in \mathbb{Z}$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

(v) Prove $x^{2n+1} + a^{2n+1}$ is divisible by $(x + a)$ for all positive integers

Prove the result is true for $n = 1$

$$x^3 + a^3$$

$$= (x+a)(x^2 - ax + a^2)$$

which is divisible by $(x + a)$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

i.e $x^{2k+1} + a^{2k+1} = (x + a)Q(x)$, where $Q(x)$ is a polynomial

Prove the result is true for $n = k + 1$

i.e Prove $x^{2k+3} + a^{2k+3} = (x + a)T(x)$, where $T(x)$ is a polynomial

Proof:

$$x^{2k+3}+a^{2k+3}=x^2\times x^{2k+1}+a^{2k+3}$$

$$=x^2\left\{(x+a)Q(x)-a^{2k+1}\right\}+a^{2k+3} \text{ (by assumption rearranged)}$$

$$=(x+a)x^2Q(x)-a^{2k+1}x^2+a^{2k+1}a^2$$

$$=(x+a)x^2Q(x)-a^{2k+1}(x^2-a^2)$$

$$=(x+a)x^2Q(x)-a^{2k+1}(x-a)(x+a)$$

$$=(x+a)\left\{x^2Q(x)-a^{2k+1}(x-a)\right\}$$

$$=(x+a)T(x), \text{ where } T(x)=x^2Q(x)-a^{2k+1}(x-a)$$

which is a polynomial

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

Exercise 2B; 2bd, 3, 4a, 5ac, 6, 7, 8a, 9, 10, 13a

in set notation: for all integers $n \geq 0 \;\Rightarrow\; \forall\, n \in \mathbb{Z} : n \geq 0$

for all even integers $n \geq 0 \Rightarrow \forall\, n \in \mathbb{Z} : n \geq 0 : \exists\, a \in \mathbb{Z} : n = 2a$