Applications of Euler's Formula

Exponential Functions

$$\begin{aligned} e^{i\theta + 2ik\pi} &= \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) \\ &= \cos\theta + i\sin\theta \text{ if } k \in \mathbb{Z} \\ &= e^{i\theta} \end{aligned}$$

$$\boxed{\begin{gathered} e^{i(\theta + 2\pi k)} = e^{i\theta} \\ \text{So } e^{z} \text{ is a periodic function, with period } 2\pi i \end{gathered}}$$

Trigonometric Functions

Earlier we proved the following identities:

$$z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta$$

rearranging these we get;

$$2\cos n\theta = e^{n\theta i} + e^{-n\theta i} \qquad 2i\sin n\theta = e^{n\theta i} - e^{-n\theta i}$$

$$\boxed{\cos n\theta = \frac{1}{2}\left(e^{n\theta i} + e^{-n\theta i}\right) \qquad \sin n\theta = \frac{1}{2i}\left(e^{n\theta i} - e^{-n\theta i}\right)}$$

e.g. Prove the identities;

(i) $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \cos^2\theta = \frac{1}{(2i)^2}\left(e^{i\theta} - e^{-i\theta}\right)^2 + \frac{1}{2^2}\left(e^{i\theta} + e^{-i\theta}\right)^2$$

$$= \frac{1}{4}\left(-e^{2i\theta} + 2 - e^{-2i\theta} + e^{2i\theta} + 2 + e^{-2i\theta}\right)$$

$$= \frac{1}{4}(4) = \underline{1}$$

$(ii)\ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$$\sin^3\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3$$

$$= \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{-8i}$$

$$= \frac{1}{4}\left(\frac{3\left(e^{i\theta} - e^{-i\theta}\right) - \left(e^{3i\theta} - e^{-3i\theta}\right)}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$(iii)\ 2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

$$2\sin\alpha\cos\beta = 2 \times \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \times \frac{e^{i\beta} + e^{-i\beta}}{2}$$

$$= \frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)} - e^{-i(\alpha+\beta)}}{2i}$$

$$= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{2i} + \frac{e^{i(\alpha-\beta)} - e^{i(\alpha-\beta)}}{2i}$$

$$= \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

Roots of Complex Numbers

e.g. Find the square roots of $1-\sqrt{3}\,i$

$$z^2 = 1-\sqrt{3}\,i$$

$$= 2e^{-\frac{i\pi}{3}}$$

$$e^z = e^{z+2i\pi k}$$

$$z = \sqrt{2}\,e^{-\frac{i\pi}{6}+\pi k},k = 0,1$$

$$z = \sqrt{2}\,e^{-\frac{i\pi}{6}},\ \sqrt{2}\,e^{\frac{5i\pi}{6}}$$

Exercise 3E; 1, 3, 4bc, 6, 7, 8b ii, iii, 11bc, 14

$$z = \sqrt{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right),\ \sqrt{2}\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$= \frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i,\ -\frac{\sqrt{6}}{2}+\frac{1}{2}i$$