Differentiating Inverse Trig

$$y = \sin^{-1} x$$
$$x = \sin y$$
$$\frac{dx}{dy} = \cos y$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$
$$= \frac{1}{\sqrt{\cos^2 y}}$$
$$= \frac{1}{\sqrt{1 - \sin^2 y}}$$
$$= \frac{1}{\sqrt{1 - x^2}}$$

$$y = \cos^{-1} x$$
$$x = \cos y$$
$$\frac{dx}{dy} = -\sin y$$
$$\frac{dy}{dx} = \frac{-1}{\sin y}$$
$$= \frac{-1}{\sqrt{\sin^2 y}}$$
$$= \frac{-1}{\sqrt{1 - \cos^2 y}}$$
$$= \frac{-1}{\sqrt{1 - x^2}}$$

$$y = \tan^{-1} x$$
$$x = \tan y$$
$$\frac{dx}{dy} = \sec^2 y$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$
$$= \frac{1}{1 + \tan^2 y}$$
$$= \frac{1}{1 + x^2}$$

In general;

$$y = \sin^{-1}\frac{x}{a} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$y = \cos^{-1}\frac{x}{a} \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$y = \tan^{-1}\frac{x}{a} \qquad \frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$y = \sin^{-1} f(x) \qquad \frac{dy}{dx} = \frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \cos^{-1} f(x) \qquad \frac{dy}{dx} = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$$

$$y = \tan^{-1} f(x) \qquad \frac{dy}{dx} = \frac{f'(x)}{1 + [f(x)]^2}$$

e.g. $(i)\ y = \sin^{-1} 5x$

$$\frac{dy}{dx} = \frac{5}{\sqrt{1 - 25x^2}}$$

$(ii)\ y = \cos^{-1} e^x$

$$\frac{dy}{dx} = \frac{-e^x}{\sqrt{1 - e^{2x}}}$$

$$(iii)\ y = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}}$$

OR

$$y = \sin^{-1}\left(\frac{x}{3}\right)$$

$$\frac{dy}{dx} = \frac{\frac{1}{3}}{\sqrt{1 - \frac{x^2}{9}}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\sqrt{9 - x^2}}$$

$$= \frac{1}{\sqrt{9 - x^2}}$$

$$(iv)\ y = e^{\cos^{-1} x}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} e^{\cos^{-1} x}$$

$$= \frac{-e^{\cos^{-1} x}}{\sqrt{1-x^2}}$$

$$(v)\ y = \left(\tan^{-1} x\right)^3$$

$$\frac{dy}{dx} = 3\left(\tan^{-1} x\right)^2 \cdot \frac{1}{1+x^2}$$

$$= \frac{3\left(\tan^{-1} x\right)^2}{1+x^2}$$

$$(vi)\ y = x^2 \tan^{-1} x^3$$

$$\frac{dy}{dx} = \left(x^2\right)\left(\frac{3x^2}{1+x^6}\right) + \left(\tan^{-1} x^3\right)(2x)$$

$$= \frac{3x^4}{1+x^6} + 2x \tan^{-1} x^3$$

(*vii*) 2018 Extension 1 HSC Question 12c) Let $f(x) = \sin^{-1} x + \cos^{-1} x$

a) Show that $f'(x) = 0$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$$

$$= 0$$

b) Hence, or otherwise, prove $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

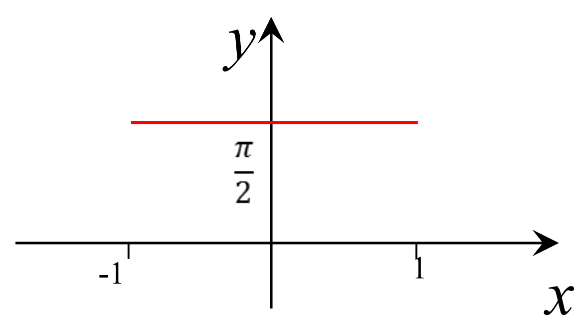
Since $f'(x) = 0$ and both $\sin^{-1} x$ and $\cos^{-1} x$ are continous functions over their domain, then $f(x)$ is a constant

When $x = 0$; $f(x) = \sin^{-1} 0 + \cos^{-1} 0$

$$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

c) Hence, sketch $f(x) = \sin^{-1} x + \cos^{-1} x$



$(viii)$ Let $f(x) = \tan^{-1} x + \tan^{-1} \dfrac{1}{x}$

a) Show that $f'(x) = 0$

$$f'(x) = \frac{1}{1+x^2} + \frac{\frac{-1}{x^2}}{1+\frac{1}{x^2}}$$

$$= \frac{1}{1+x^2} + \frac{-1}{x^2+1} = 0$$

b) Sketch $y = f(x)$

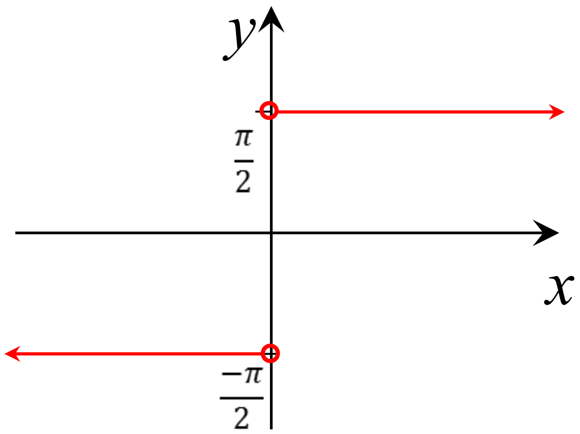
$\tan^{-1}\dfrac{1}{x}$ is discontinous at $x = 0$

When $x = 1$; $f(x) = \tan^{-1} 1 + \tan^{-1} 1$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

When $x = -1$; $f(x) = \tan^{-1} -1 + \tan^{-1} -1$

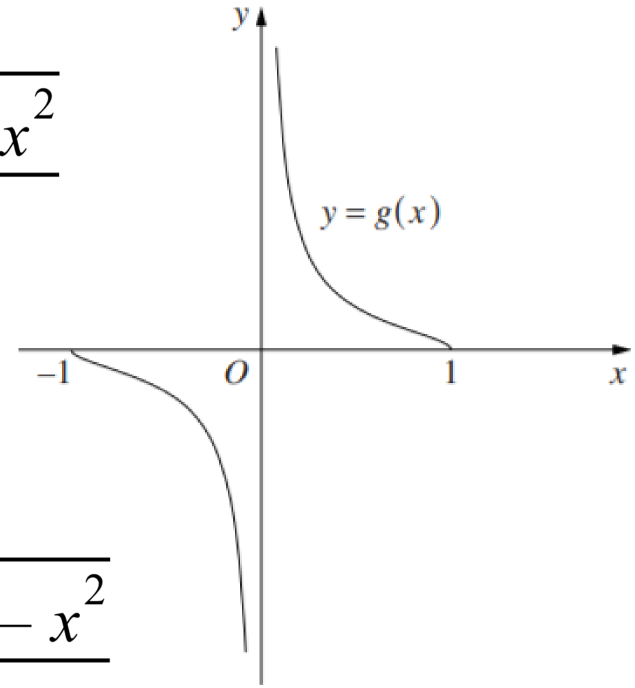
$$= \frac{-\pi}{4} + \frac{-\pi}{4} = -\frac{\pi}{2}$$



($viii$) **2020 Extension 1 HSC Question 13c)**

Suppose $f(x) = \tan(\cos^{-1}(x))$ and $g(x) = \dfrac{\sqrt{1-x^2}}{x}$

The graph of $y = g(x)$ is given



(i) Show that $f'(x) = g'(x)$

$$f(x) = \tan(\cos^{-1}(x))$$

$$f'(x) = \frac{-\sec^2(\cos^{-1}(x))}{\sqrt{1-x^2}}$$

$$= \frac{-1}{\cos^2(\cos^{-1}(x))\sqrt{1-x^2}}$$

$$= \frac{-1}{x^2\sqrt{1-x^2}}$$

$$g(x) = \frac{\sqrt{1-x^2}}{x}$$

$$g'(x) = \frac{(x)\left(\dfrac{-x}{\sqrt{1-x^2}}\right) - \left(\sqrt{1-x^2}\right)(1)}{x^2}$$

$$= \frac{-x^2 - 1 + x^2}{x^2\sqrt{1-x^2}}$$

$$= \frac{-1}{x^2\sqrt{1-x^2}}$$

thus $f'(x) = g'(x)$

(ii) Using part (i), or otherwise, show that $f(x) = g(x)$

$$f'(x) - g'(x) = 0$$

$$\int (f'(x) - g'(x)) = c$$

$$f(x) - g(x) = c$$

$$f(x) = g(x) + c$$

$\underline{x > 0}: f(1) - g(1) = c$

$$\tan(\cos^{-1}(1)) = \frac{\sqrt{1 - 1^2}}{1} +$$

$$\tan 0 = 0 + c$$

$$c = 0$$

$\underline{x < 0}: f(-1) - g(-1) = c$

$$\tan(\cos^{-1}(-1)) = \frac{\sqrt{1 - (-1)^2}}{-1} + c$$

$$\tan \pi = 0 + c$$

$$c = 0$$

$$\text{thus } f(x) = g(x) + 0$$

$$\underline{\text{i.e. } f(x) = g(x)}$$

Exercise 12A; 2ace etc, 3bd, 4a, 5, 6a, 9ace etc, 11, 13, 15, 16, 18, 20