The Language of Logic

Proposition: a sentence proposing an idea that can be true (T) or false (F), but not both.

e.g. "it is raining" is a proposition

"is it raining" is NOT a proposition

Logic has its own language with an alphabet consisting of;

(i) propositional variables: e.g. p_1, p_2, ..., q , X , Y, Φ

(ii) punctuation symbols: the two parentheses i.e. (and)

(iii) logical symbols: connectives

$\Rightarrow$ implication; used for "if then" statements

$\neg$ (or $\sim$) negation; the logical complement to a proposition

$\Leftrightarrow$ equivalence; propositions that are logically equivalent

$\wedge$ and; conjunction of two propositions

$\vee$ or; disjunction of two propositions

(iv) quantifiers: $\exists$ (there exists) and $\forall$ (for all)

Statements are;

(i) propositions *OR*

(ii) a series of propositions connected by punctuation symbols or logical symbols

A logically valid argument: a list of propositions from which a conclusion follows.

This is also called a **proof**

example: a situation that demonstrates the assertion of a proposition

counterexample: an example that demonstrates that a proposition is not true in general

It is used to invalidate an argument

Negation

The negation of a statement changes its logical value i.e. T to F or F to T

$$\neg(X \wedge Y) \iff \neg X \vee \neg Y$$

$$\neg(X \vee Y) \iff \neg X \wedge \neg Y$$

$$\neg\{\forall x, A(x)\} \iff \exists x : \neg A(x)$$

$$\neg\{\exists x : A(x)\} \iff \forall x\ ,\ \neg A(x)$$

e.g. (i) X: the number is even
(6 would be T)

$\neg$X: the number is not even
(6 would be F)

(ii) Y: all birds are black

$\neg$Y: at least one bird is not black
or not all birds are black
or some birds are not black

(iii) Z: $\exists\ x \in \mathbb{R} : x^2 < 0$
(there exists a real number such that its square is negative)

$\neg$Z: $\forall\ x \in \mathbb{R}\ ,\ x^2 \geq 0$
(for all real numbers, their squares are greater than or equal to zero)

Implication

(if then statements)

$$P \Rightarrow Q$$

If P is true, then Q must also be true i.e P implies Q

P is the **premise** of the implication, and Q is the **conclusion**

P is a **sufficient** condition for Q

Q can still exist without P, however if P exists Q must exist

Q is a **necessary** condition for P

P cannot exist unless Q exists

e.g. X: you score 90% or above in Extension 2

Y: you always listen to your teacher

$$X \Rightarrow Y$$

Scoring 90% or above is sufficient to conclude that you always listen to your teacher

It is necessary to listen to your teacher in order to score 90% or above in Extension 2

Note: $\neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q$

Converse

To find the converse of an implication, reverse the implication

the converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

The converse of a true statement may or may not be true. Similarly the converse of a false statement may or may not be false

e.g. X: it is raining
Y: the grass is wet

Whilst it may be true that $X \Rightarrow Y$,
it is not necessarily true that $Y \Rightarrow X$

Equivalence

(if and only if statements, iff)

$P \Leftrightarrow Q$

Two statements are equivalent if each is the consequence of the other

e.g. X: *ABCD* is a parallelogram
Y: diagonals *AC* and *BD* bisect

$(X \Rightarrow Y) \wedge (Y \Rightarrow X)$

$\therefore X \Leftrightarrow Y$

ABCD is a parallelogram iff the diagonals bisect each other

Contrapositive

The contrapositive of an implication is formed when you negate the converse.

original statement: $P \Rightarrow Q$

contrapositive statement: $\neg Q \Rightarrow \neg P$

$$P \Rightarrow Q \Leftrightarrow \neg Q \Rightarrow \neg P$$

An implication is logically equivalent to its contrapositive

e.g. X: a polygon is a triangle
Y: interior angle sum is 180º

$X \Rightarrow Y$: if a polygon is a triangle then the interior angle sum is 180º

$\neg Y \Rightarrow \neg X$: if the interior angle sum is not 180º then the polygon is not a triangle

$$X \Rightarrow Y \Leftrightarrow \neg Y \Rightarrow \neg X$$

Truth Tables

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**

P: the cat is white Q: the dog is black

e.g. Alfred, Kurt and Rudolf are accused of stealing and selling copies of an exam. At the inquiry, they testify as follows;

Alfred: Kurt is guilty and Rudolf is innocent
Kurt: If Alfred is guilty, then so is Rudolf
Rudolf: I am innocent, but at least one of the others is guilty

(i) can everyone be telling the truth?

(ii) if everyone is innocent, who lied?

(iii) Assuming everyone's testimony is true, who is guilty?

(iv) If the innocent tell the truth and the guilty lie, who is guilty?

A: Alfred is guilty K: Kurt is guilty R: Rudolf is guilty

			Alfred			Kurt			Rudolf				
A	K	R	K	$\wedge$	$\neg R$	A	$\Rightarrow$	R	$\neg R$	$\wedge$	$($ A	$\vee$	$K)$
T	T	T	T	F	F	T	T	T	F	F	T	T	T
T	T	F	T	T	T	T	F	F	T	T	T	T	T
T	F	T	F	F	F	T	T	T	F	F	T	T	F
T	F	F	F	F	T	T	F	F	T	T	T	T	F
F	T	T	T	F	F	F	T	T	F	F	F	T	T
F	T	F	T	T	T	F	T	F	T	T	F	T	T
F	F	T	F	F	F	F	T	T	F	F	F	F	F
F	F	F	F	F	T	F	T	F	T	F	F	F	F

(i) can everyone be telling the truth? *Yes*

(ii) if everyone is innocent, who lied? *Alfred and Rudolf*

(iii) Assuming everyone's testimony is true, who is guilty? *Kurt*

(iv) If the innocent tell the truth and the guilty lie, who is guilty?

Alfred and Rudolf

Exercise 2A;1, 2acef, 3, 4acdgh, 5bdf, 6ad, 7, 8, 9acf, 10, 11, 12a, 13, 15, 16