

Algebraic Inequalities

Properties of Inequalities

If $a > b$ then $a \pm x > b \pm x$

you can add or subtract the same amount to both sides

If $a > b$ then $ax > bx$ for $x > 0$

you can multiply or divide both sides by a positive amount

If $a > b$ then $ax < bx$ for $x < 0$

the inequality sign changes when you multiply or divide both sides by a negative amount

If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$

when both sides are positive, the inequality sign changes when you take the reciprocal of both sides

If $a > b > 0$ then $a^2 > b^2$

when both sides are positive, you can square both sides

If $a > b$ and $b > c$ then $a > c$

inequalities of the same sign can be linked

If $a > b > 0$ and $c > d > 0$
then $ab > cd$

when both sides are positive, inequalities of the same sign can be multiplied

If $a > b > 0$ and $c > d > 0$
then $a + c > b + d$

when both sides are positive, inequalities of the same sign can be added

Inequality Techniques

To prove $x \geq y$, it can be easier to prove $x - y \geq 0$

e.g. (i) (1995) Prove $pq \leq \frac{p^2 + q^2}{2}$

$$\begin{aligned}\frac{p^2 + q^2}{2} - pq &= \frac{p^2 - 2pq + q^2}{2} \\ &= \frac{(p - q)^2}{2} \\ &\geq 0\end{aligned}$$

$$\therefore \underline{\frac{p^2 + q^2}{2} \geq pq}$$

OR Assume $pq > \frac{p^2 + q^2}{2}$

$$2pq > p^2 + q^2$$

$$0 > p^2 - 2pq + q^2$$

$$0 > (p - q)^2$$

But $(p - q)^2 > 0$ which is a contradiction

$$\therefore \underline{pq \leq \frac{p^2 + q^2}{2}}$$

Start with a known result

(ii) (1994) a) Prove $a^2 + b^2 + c^2 > ab + bc + ac$

$$(a - b)^2 > 0$$

$$a^2 - 2ab + b^2 > 0$$

$$\therefore a^2 + b^2 > 2ab$$

$$a^2 + c^2 > 2ac$$

$$b^2 + c^2 > 2bc$$

$$2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc$$

$$\underline{a^2 + b^2 + c^2 > ab + ac + bc}$$

b) If $a + b + c = 1$, prove $ab + ac + bc < \frac{1}{3}$

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + ac + bc)$$

$$\therefore (a + b + c)^2 - 2(ab + ac + bc) > ab + ac + bc$$

$$3(ab + ac + bc) < (a + b + c)^2$$

$$3(ab + ac + bc) < 1$$

$$\underline{ab + ac + bc < \frac{1}{3}}$$

Substitute different expressions into known inequalities

c) Prove $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$

$$a^2 + b^2 + c^2 \geq ab + ac + bc$$

$$a^2 + b^2 + c^2 - ab - ac - bc \geq 0$$

$$(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) \geq 0$$

$$a^3 + ab^2 + ac^2 - a^2b - a^2c - abc + a^2b + b^3 + bc^2 - ab^2 - abc - b^2c \\ + a^2c + b^2c + c^3 - abc - ac^2 - bc^2 \geq 0$$

$$a^3 + b^3 + c^3 - 3abc \geq 0$$

$$\frac{1}{3}(a^3 + b^3 + c^3) \geq abc$$

let $a = a^{\frac{1}{3}}, b = b^{\frac{1}{3}}, c = c^{\frac{1}{3}}$

$$\frac{1}{3}(a+b+c) \geq a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

Arithmetic Mean \geq Geometric Mean

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

d) Suppose $(1+x)(1+y)(1+z) = 8$, prove $xyz \leq 1$

$$(1+x)(1+y)(1+z) = 8$$

$$1 + x + y + xy + z + xz + yz + xyz = 8$$

$$\frac{1}{3}(x + y + z) \geq \sqrt[3]{xyz} \quad \text{AM} \geq \text{GM}$$

$$\underline{x + y + z \geq 3\sqrt[3]{xyz}}$$

$$xy + yz + xz \geq 3\sqrt[3]{(xy)(yz)(xz)}$$

$$xy + yz + xz \geq 3\sqrt[3]{x^2 y^2 z^2}$$

$$\underline{xy + yz + xz \geq 3\left(\sqrt[3]{xyz}\right)^2}$$

$$1 + x + y + z + xy + xz + yz + xyz = 8$$

$$1 + 3\sqrt[3]{xyz} + 3\left(\sqrt[3]{xyz}\right)^2 + xyz \leq 8$$

$$1 + 3\sqrt[3]{xyz} + 3\left(\sqrt[3]{xyz}\right)^2 + \left(\sqrt[3]{xyz}\right)^3 \leq 8$$

$$\left(1 + \sqrt[3]{xyz}\right)^3 \leq 8$$

$$1 + \sqrt[3]{xyz} \leq 2$$

$$\sqrt[3]{xyz} \leq 1$$

$$\underline{xyz \leq 1}$$

OR

$$(1+x) \geq 2\sqrt{x}$$

AM \geq GM

$$(1+y) \geq 2\sqrt{y}$$

$$(1+z) \geq 2\sqrt{z}$$

$$(1+x)(1+y)(1+z) \geq 2\sqrt{x} \times 2\sqrt{y} \times 2\sqrt{z}$$

$$= 8\sqrt{xyz}$$

$$\therefore 8 \geq 8\sqrt{xyz}$$

$$1 \geq \sqrt{xyz}$$

$$\sqrt{xyz} \leq 1$$

$$\underline{xyz \leq 1}$$

(iii) Prove $\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 2\sqrt{ab} \times 2\sqrt{\frac{1}{ab}} \quad (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 3\sqrt[3]{abc} \times 3\sqrt[3]{\frac{1}{abc}}$$

$$\frac{1}{a} + \frac{1}{b} \stackrel{=4}{\geq} \frac{4}{a+b}$$

$$\frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c}$$

$$\frac{1}{a} + \frac{1}{c} \geq \frac{4}{a+c}$$

$$\frac{2}{a} + \frac{2}{b} + \frac{2}{c} \geq \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{a+c}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c}$$

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \stackrel{=9}{\geq} \frac{9}{a+b+c}$$

Replace a with $a + b$

b with $b + c$

c with $a + c$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{a+c} \geq \frac{9}{2(a+b+c)}$$

$$\frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \geq \frac{9}{a+b+c}$$

$$\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

(iv) a) If $a > 0$ is a real number, show that $a + \frac{1}{a} \geq 2$

$$a + \frac{1}{a} \geq 2\sqrt{a \times \frac{1}{a}} \quad \text{AM} \geq \text{GM} \quad \text{OR} \quad \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$$

$$a + \frac{1}{a} \geq 2\sqrt{1} \qquad a - 2 + \frac{1}{a} \geq 0$$

$$\underline{a + \frac{1}{a} \geq 2} \qquad a + \frac{1}{a} \geq 2$$

If $a > 0$, $b > 0$ and $c > 0$;

b) Show that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$

$$\frac{b}{a} + \frac{a}{b} \geq 2$$

$$\frac{c}{a} + \frac{a}{c} \geq 2$$

$$\frac{c}{b} + \frac{b}{c} \geq 2$$

$$\frac{b}{a} + \frac{a}{b} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{c} \geq 6$$

$$\underline{\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6}$$

c) Show that $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Replace a with $b + c$

b with $a + c$

c with $a + b$

$$\frac{a+c+a+b}{b+c} + \frac{a+b+b+c}{a+c} + \frac{b+c+a+c}{a+b} \geq 6$$

$$\frac{2a}{b+c} + 1 + \frac{2b}{a+c} + 1 + \frac{2c}{a+b} + 1 \geq 6$$

$$\frac{2a}{b+c} + \frac{2b}{a+c} + \frac{2c}{a+b} \geq 3$$

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

**Exercise 2D; 2b, 4b, 5a, 8, 11, 12,
13, 14, 15, 16, 19, 20**