Combinations

A combination is a set of objects where the order that they are arranged is not important.
i.e. a **selection**

If we arrange objects in a line, and the order is not important then;

A B is the same selections as **B A**

e.g. 5 objects, arrange 2 of them

A B	~~**B A**~~	~~**C A**~~	~~**D A**~~	~~**E A**~~
A C	**B C**	~~**C B**~~	~~**D B**~~	~~**E B**~~
A D	**B D**	**C D**	~~**D C**~~	~~**E C**~~
A E	**B E**	**C E**	**D E**	~~**E D**~~

$$\text{Permutations} = {}^5P_2$$
$$= 20$$

$$\text{Combinations} = \frac{20}{2!}$$
$$= 10$$

5 objects, arrange 3 of them

A B C	**B A C**	**C A B**	**D A B**	**E A B**
A B D	**B A D**	**C A D**	**D A C**	**E A C**
A B E	**B A E**	**C A E**	**D A E**	**E A D**
A C B	**B C A**	**C B A**	**D B A**	**E B A**
A C D	**B C D**	**C B D**	**D B C**	**E B C**
A C E	**B C E**	**C B E**	**D B E**	**E B D**
A D B	**B D A**	**C D A**	**D C A**	**E C A**
A D C	**B D C**	**C D B**	**D C B**	**E C B**
A D E	**B D E**	**C D E**	**D C E**	**E C D**
A E B	**B E A**	**C E A**	**D E A**	**E D A**
A E C	**B E C**	**C E B**	**D E B**	**E D B**
A E D	**B E D**	**C E D**	**D E C**	**E D C**

$$\text{Permutations} = {}^5P_3$$
$$= 60$$

$$\text{Combinations} = \frac{60}{3!}$$
$$= 10$$

If we have n different objects, and we arrange k of them and are not concerned about the order;

$$\text{Number of Arrangements} = \frac{{}^{n}P_{k}}{k!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$\boxed{= {}^{n}C_{k}}$$

e.g. (i) How many ways can 6 numbers be chosen from 45 numbers?

$$\text{Ways} = {}^{45}C_{6}$$

$$\underline{= 8145060}$$

Note: at 70 cents per game, \$5 701 542 = amount of money you have to spend to guarantee a win in Lotto.

(*ii*) Committees of five people are to be obtained from a group of seven men and four women.

How many committees are possible if;

a) there are no restrictions?

$$\text{Committees} = {}^{11}C_5$$
$$\underline{= 462}$$

With no restrictions, choose 5 people from 11, gender does not matter

b) the committee contains only males?

$$\text{Committees} = {}^{7}C_5$$
$$\underline{= 21}$$

By restricting it to only males, there is only 7 people to choose from

c) the committee contains at least one woman?

$$\text{Committees} = 462 - 21$$
$$\underline{= 441}$$

easier to work out male only and subtract from total number of committees

(iii) A hand of five cards is dealt from a regular pack of fifty two cards.

a) What is the number of possible hands?

$$\begin{aligned}\text{Hands} &= {}^{52}C_5 \\ &= \underline{2598960}\end{aligned}$$

b) What is the probability of getting "three of a kind"?

choose which number has "three of a kind"

choose three of those cards

choose remaining two cards from the rest

$$\begin{aligned}\text{Hands} &= {}^{13}C_1 \times {}^{4}C_3 \times {}^{48}C_2 \\ &= \underline{58656}\end{aligned}$$

$$\begin{aligned}P(\text{three of a kind}) &= \frac{58656}{2598960} \\ &= \underline{\frac{94}{2915}}\end{aligned}$$

(=3.2%)

2004 Extension 1 HSC Q2e)

A four person team is to be chosen at random from nine women and seven men.

(i) In how many ways can this team be chosen?

$$\begin{aligned}\text{Teams} &= {}^{16}C_4 \\ &= \underline{1820}\end{aligned}$$

With no restrictions, choose 4 people from 16, gender does not matter

(ii) What is the probability that the team will consist of four women?

$$\begin{aligned}\text{Teams} &= {}^{9}C_4 \\ &= 126\end{aligned}$$

By restricting it to only women, there is only 9 people to choose from

$$\begin{aligned}P(4\text{ women team}) &= \frac{126}{1820} \\ &= \underline{\frac{9}{130}}\end{aligned}$$

Dividing into Groups

Case 1: dividing a group of (m + n) objects into two groups containing m objects and n objects

$$\text{Number of selections} = {}^{m+n}C_m$$

$$\boxed{= \frac{(m+n)!}{m!n!}}$$

Case 2: dividing a group of (2m) objects into two groups each containing m objects

$$\text{Number of selections} = \frac{{}^{2m}C_m}{2!}$$

$$\boxed{= \frac{(2m)!}{m!m!2!}}$$

Divide by the number of ways the groups of same size can be arranged

Case 3: dividing a group of (*m* + *n* + *p*) objects into two groups containing *m* objects, *n* objects and *p* objects respectively

$$\text{Number of selections} = {}^{m+n+p}C_m \times {}^{n+p}C_n$$
$$= \frac{(m+n+p)!}{m!(n+p)!} \times \frac{(n+p)!}{n!\,p!}$$
$$\boxed{= \frac{(m+n+p)!}{m!\,n!\,p!}}$$

e.g. (*i*) In how many ways can 15 students be divided into three equal groups?

$$\text{Ways} = \frac{15!}{5!5!5!\,3!}$$
$$= \underline{126126}$$

Divide by three as groups are of the same size

(ii) A restaurant has 3 tables, each with 4 seats. 12 people arrive for dinner. How many ways can they assigned to the tables?

$$\text{Ways} = \frac{12!}{4!4!4!3!} \times 3!$$

Divide by patrons into three groups

Now the groups have to be arranged, so they can be assigned to the tables

$$= \frac{12!}{4!4!4!}$$

$$= \underline{34650}$$

If the groups are going to be arranged, there is no need to divide by the arrangements!!!!

A permutation is a combination multiplied by the number of ways the objects can be arranged

i.e. **select then arrange**

Insertion Method

Useful when some objects are NOT allowed to be together

e.g. The letters of the word **BETWEEN** are arranged in a line. In how many ways can they be arranged if;

(i) All the **E**'s are separated?

Traditional method: *work out the number of ways the E's can be placed*

___	___	___	___	___	___	___	
E		**E**		**E**	**E**	**E**	3
E			**E**		**E**	**E**	2
E				**E**		**E**	1
	E		**E**		**E**	**E**	2
	E			**E**		**E**	1
		E		**E**		**E**	1
							10

$$\text{Ways} = 10 \times 4!$$

$$= \underline{240}$$

***Insertion method**: place the unrestricted objects first, creating spots for the E's to be placed.*

____ B ____ T ____ W ____ N ____

Once **B, T, W** and **N** have ben arranged there are now 5 spaces that the **E**'s can go (note: order not important as objects are the same)

$$\text{Ways} = 4! \times {}^{5}C_{3}$$
$$= \underline{240}$$

(ii) Exactly two of the **E**'s are together?

(note: this time order is important as objects are not the same **EE** and **E**)

$$\text{Ways} = 4! \times {}^{5}P_{2}$$
$$= \underline{480}$$

Using Separators

Useful when dividing large groups into smaller groups

e.g. Three pirates are sharing out the contents of a treasure chest containing forty-eight gold coins and two lead coins.

The first pirate takes out coins one at a time until a lead coin is taken.

The second pirate then takes out coins one at a time until the second lead coin is taken.

The third pirate then takes all of the remaining coins.

In how many ways can the coins be distributed?

The question is equivalent to how many ways can 2 L's and 48 G's be arranged. (The 2 L's act as separators of the 3 pirates)

$$\text{Ways} = \frac{50!}{48!2!}$$

$$= \underline{1225}$$

2012 Extension 2 HSC Question 16 a) (ii)

In how many ways can 10 identical coins be allocated to 4 different boxes?

The question is equivalent to how many ways can 3 S's and 10C's be arranged. (The 3 S's act as separators of the 4 boxes)

$$\text{Ways} = \frac{13!}{3!10!}$$

$$= \underline{286}$$

Note: 1% of the state got this correct!!!

2013 Extension 2 HSC Question 10

A hostel has four vacant rooms. Each room can accommodate a maximum of four people.

In how many ways can six people be accommodated in the four rooms?

$$\text{Total Ways no restrictions} = 4^6$$
$$= 4096$$

Each person has a choice of 4 rooms

$$\text{Less ways with 6 in a room} = {}^4C_1$$
$$= 4$$

Choose which of the 4 rooms will have the six people

$$\text{Less ways with 5 in a room} = {}^4C_1 \times {}^6C_5 \times {}^3C_1$$
$$= 72$$

Choose which of the 4 rooms will have the five people, then choose the five people to go in that room, then which of the remaining rooms will have one

$$\text{Ways } = 4096 - 4 - 72$$
$$= \underline{4020}$$

Note: 15% of the state got this correct!!!
It was multiple choice

Exercise 14E; 1, 3, 5, 7, 9, 11, 13, 14, 17, 19, 21, 22, 23c, 25a, 26, 28, 29