Reduction Formula

Reduction (or recurrence) formulae can be used when the integrand is raised to a power.

Integration by parts often used to find the formula.

e.g. (i) Find $\int_1^e x(\log x)^3\,dx$

$$u = (\log x)^n \qquad v = \frac{1}{2}x^2$$

$$du = \frac{n(\log x)^{n-1}}{x}dx \qquad dv = x\,dx$$

$$I_n = \int_1^e x(\log x)^n\,dx$$

$$= \frac{1}{2}\left[x^2(\log x)^n\right]_1^e - \frac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\log x)^{n-1}\,dx$$

$$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$$

$$\therefore \int_1^e x(\log x)^3\,dx = I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$$

$$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - I_1\right)$$

$$= -\frac{e^2}{4} + \frac{3}{2} I_1$$

$$= -\frac{e^2}{4} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2} I_0\right)$$

$$= \frac{e^2}{2} - \frac{3}{4} I_0$$

$$= \frac{e^2}{2} - \frac{3}{4}\int_1^e x\,dx$$

$$= \frac{e^2}{2} - \frac{3}{8}\left[x^2\right]_1^e$$

$$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8} \quad = \frac{e^2}{8} + \frac{3}{8}$$

OR

$$I_0 = \int_1^e x\,dx = \frac{1}{2}\left[x^2\right]_1^e$$

$$= \frac{1}{2}\left(e^2 - 1\right)$$

$$I_1 = \frac{e^2}{2} - \frac{1}{2} I_0 = \frac{e^2}{2} - \frac{1}{2} \times \frac{1}{2}\left(e^2 - 1\right)$$

$$= \frac{e^2}{4} + \frac{1}{4}$$

$$I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{2} - \frac{1}{4}\left(e^2 + 1\right)$$

$$= \frac{e^2}{4} - \frac{1}{4}$$

$$I_3 = \frac{e^2}{2} - \frac{3}{2} I_2 = \frac{e^2}{2} - \frac{3}{2} \times \frac{1}{4}\left(e^2 - 1\right)$$

$$= \frac{e^2}{8} + \frac{3}{8}$$

Integration by parts is the commonest way of getting reduction formulae, but it is not the only method.

Some just involve the use of a **trig identity**.

(ii) Given that $I_n = \int \cot^n x dx$, find I_6

$$I_n = \int \cot^n x dx$$

$$= \int \cot^{n-2} x \cot^2 x dx$$

$$= \int \cot^{n-2} x \left(\operatorname{cosec}^2 x - 1\right) dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x dx - \int \cot^{n-2} x dx$$

$$= -\int u^{n-2} du - I_{n-2}$$

$$= -\frac{1}{n-1} u^{n-1} - I_{n-2}$$

$$= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2}$$

$\int \cot^n x dx$

Uses the same technique, for all powers.

So just use that technique

$$u = \cot x$$

$$du = -\operatorname{cosec}^2 x dx$$

OR

Reduction formulae involving trig often reduces by 2. If you do not have to use parts you could try $I_n \pm I_{n+2}$

$$I_n + I_{n+2} = \int \cot^n x dx + \int \cot^{n+2} x dx$$

$$= \int \left(\cot^n x + \cot^{n+2} x \right) dx$$

$$= \int \left\{ \cot^n x \left(1 + \cot^2 x \right) \right\} dx$$

$$= \int \cot^n x \operatorname{cosec}^2 x dx$$

$$= -\frac{\cot^{n+1} x}{n+1}$$

$$I_{n+2} = -\frac{\cot^{n+1} x}{n+1} - I_n$$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\int \cot^6 x dx = I_6$$

$$= -\frac{1}{5}\cot^5 x - I_4$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x + I_2$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - I_0$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - \int dx$$

$$= -\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + c$$

Sometimes integration by parts is not enough by itself.

Some involve the use of **parts** along with **algebraic manipulation** or use of a **trig identity**.

(iii) Given that $I_n = \int (1+x^2)^n \, dx,$ find a reduction formula connecting I_n with I_{n-1}

$u = (1+x^2)^n \qquad v = x$

$du = 2nx(1+x^2)^{n-1} dx \qquad dv = dx$

$$I_n = \int (1+x^2)^n \, dx$$

$$= x(1+x^2)^n - 2n\int x^2 (1+x^2)^{n-1} dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2-1)(1+x^2)^{n-1} dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2)(1+x^2)^{n-1} dx + 2n\int (1+x^2)^{n-1} dx$$

$$= x(1+x^2)^n - 2n\int (1+x^2)^n dx + 2n\int (1+x^2)^{n-1} dx$$

$$= x(1+x^2)^n - 2nI_n + 2nI_{n-1}$$

$$(2n+1)I_n = x(1+x^2)^n + 2nI_{n-1}$$

$$\therefore I_n = \frac{x(1+x^2)^n}{(2n+1)} + \frac{2n}{(2n+1)} I_{n-1}$$

e.g. (*iv*) (1987)

Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, prove that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

where n is an integer and $n \geq 2$, hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x dx$

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$u = \cos^{n-1} x \qquad v = \sin x$$

$$du = -(n-1)\cos^{n-2} x \sin x dx \qquad dv = \cos x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx$$

$$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$$

$$= \left\{\cos^{n-1}\frac{\pi}{2}\sin\frac{\pi}{2} - \cos^{n-1} 0 \sin 0\right\} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$\therefore nI_n = (n-1)I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right)I_{n-2}$$

OR

$$I_{n-2} - I_n$$

$$= \int_0^{\frac{\pi}{2}} \left(\cos^{n-2} x - \cos^n x \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x \left(1 - \cos^2 x \right) dx$$

$$u = \sin x \qquad v = \frac{-\cos^{n-1} x}{n-1}$$

$$du = \cos x dx \qquad dv = \cos^{n-2} x \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$$

$$= \frac{-1}{n-1} \left[\sin x \cos^{n-1} x \right]_0^{\frac{\pi}{2}} + \frac{1}{n-1} \int_0^{\frac{\pi}{2}} \cos^n dx$$

$$= \frac{1}{n-1} I_n$$

$$\therefore (n-1) I_{n-2} - (n-1) I_n = I_n$$

$$(n-1) I_{n-2} = n I_n$$

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x dx = I_5$$

$$= \frac{4}{5} I_3$$

$$= \frac{4}{5} \times \frac{2}{3} I_1$$

$$= \frac{8}{15} \int_0^{\frac{\pi}{2}} \cos x dx$$

$$= \frac{8}{15} [\sin x]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15} \left(\sin \frac{\pi}{2} - \sin 0 \right)$$

$$= \frac{8}{15}$$

(*v*) (2004 Question 8b)

Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2, \dots$

a) Show that $I_n + I_{n+2} = \dfrac{1}{n+1}$

$$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n dx + \int_0^{\frac{\pi}{4}} \tan^{n+2} dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x\left(1 + \tan^2 x\right) dx$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x dx$$

when $x = 0, u = 0$

$$x = \frac{\pi}{4}, u = 1$$

$$= \int_0^1 u^n du$$

$$= \left[\frac{u^{n+1}}{n+1}\right]_0^1$$

$$= \frac{1}{n+1} - 0 \quad = \underline{\frac{1}{n+1}}$$

$b)$ Deduce that $J_n - J_{n-1} = \dfrac{(-1)^n}{2n-1}$ for $n \geq 1$

$$J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$$

$$= (-1)^n I_{2n} + (-1)^n I_{2n-2}$$

$$= (-1)^n \left(I_{2n} + I_{2n-2} \right)$$

$$= \frac{(-1)^n}{2n-2+1} \qquad \text{(using part a) let } n = 2n - 2\text{)}$$

$$= \frac{(-1)^n}{2n-1}$$

$c)$ Show that $J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$

$$J_m - J_{m-1} = \frac{(-1)^m}{2m-1}$$

$$J_m = \frac{(-1)^m}{2m-1} + J_{m-1}$$

$$= \frac{(-1)^m}{2m-1} + \frac{(-1)^{m-1}}{2m-3} + J_{m-2}$$

$$= \frac{(-1)^m}{2m-1} + \frac{(-1)^{m-1}}{2m-3} + \ldots + \frac{(-1)^1}{1} + J_0$$

$$= \sum_{n=1}^{m} \frac{(-1)^n}{2n-1} + \int_0^{\frac{\pi}{4}} dx$$

$$= \sum_{n=1}^{m} \frac{(-1)^n}{2n-1} + [x]_0^{\frac{\pi}{4}}$$

$$= \sum_{n=1}^{m} \frac{(-1)^n}{2n-1} + \frac{\pi}{4}$$

$d)$ Use the substitution $u = \tan x$ to show that $I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$$

$$u = \tan x \Rightarrow x = \tan^{-1} u$$

$$I_n = \int_0^1 u^n \times \frac{du}{1+u^2}$$

$$dx = \frac{du}{1+u^2}$$

$$I_n = \int_0^1 \frac{u^n\,du}{1+u^2}$$

when $x = 0, u = 0$

$$x = \frac{\pi}{4}, u = 1$$

$e)$ Deduce that $0 \le I_n \le \frac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$

$$\frac{u^n}{1+u^2} \ge 0, \text{ for all } u \ge 0$$

$$\therefore I_n = \int_0^1 \frac{u^n}{1+u^2}\,du \ge 0, \text{ for all } u \ge 0$$

$$I_n + I_{n+2} = \frac{1}{n+1}$$

$$I_n = \frac{1}{n+1} - I_{n+2}$$

$$\therefore I_n \leq \frac{1}{n+1} \text{ , as } I_{n+2} \geq 0$$

$$\therefore 0 \leq I_n \leq \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{1}{n+1} = 0$$

$$\therefore \lim_{n\to\infty} I_n = 0$$

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty} (-1)^n I_{2n}$$

$$= 0$$

Exercise 2G; 2, 5, 6, 7, 8, 9, 10, 12, 14, 17