Ordinary Annuity

The difference between an annuity due and an ordinary annuity is the deposit/payment is made at the end of the compounding period instead of the start.

$$FV = P + PR + PR^2 + \ldots + PR^{n-1}$$

$$= \frac{P(R^n - 1)}{R - 1}$$

$$(PV)R^n = FV$$

$$PV = \frac{P\left(1 - R^{-n}\right)}{R - 1}$$

$P =$ principal

$R = 1 +$ interest rate as a decimal(or fraction)

$n =$ time periods

Loan Repayments

The amount still owing after n time periods is;

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$

e.g. (i) Richard and Kathy borrow \$20000 from the bank to go on an overseas holiday. Interest is charged at 12% p.a., compounded monthly. They start repaying the loan one month after taking it out, and their monthly instalments are \$300.

a) How much will they still owe the bank at the end of six years?

Initial loan is borrowed for 72 months $= 20000(1.01)^{72}$

1st repayment invested for 71 months $= 300(1.01)^{71}$

2nd repayment invested for 70 months $= 300(1.01)^{70}$

$\vdots$

2nd last repayment invested for 1 month $= 300(1.01)^{1}$

last repayment invested for 0 months $= 300$

Repayments are an investment in your loan

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$

$$A_{72} = 20000(1.01)^{72} - \left\{300 + 300(1.01) \ldots + 300(1.01)^{70} + 300(1.01)^{71}\right\}$$

$$a = 300, r = 1.01, n = 72$$

$$= 20000(1.01)^{72} - \left\{\frac{a(r^n - 1)}{r - 1}\right\}$$

$$= 20000(1.01)^{72} - \left\{\frac{300(1.01^{72} - 1)}{0.01}\right\}$$

$$= \underline{\$9529.01}$$

b) How much interest will they have paid in six years?

$$\text{Total repayments} = 300 \times 72$$

$$= \$21600$$

$$\text{Loan reduction} = 20000 - 9529.01$$

$$= \$10470.99$$

$$\therefore Interest = 21600 - 10470.99$$

$$= \underline{\$11129.01}$$

(ii) *Finding the amount of each instalment*

Yog borrows $30000 to buy a car. He will repay the loan in five years, paying 60 equal monthly instalments, beginning one month after he takes out the loan. Interest is charged at 9% p.a. compounded monthly.

Find how much the monthly instalment shold be.

Let the monthly instalment be $M

Initial loan is borrowed for 60 months $= 30000(1.0075)^{60}$

1^{st} repayment invested for 59 months $= M(1.0075)^{59}$

2^{nd} repayment invested for 58 months $= M(1.0075)^{58}$

$\vdots$

2^{nd} last repayment invested for 1 month $= M(1.0075)^{1}$

last repayment invested for 0 months $= M$

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$

$$A_{60} = 30000(1.0075)^{60} - \left\{M + M(1.0075) + \ldots + M(1.0075)^{58} + M(1.0075)^{59}\right\}$$

$$a = M, r = 1.0075, n = 60$$

$$= 30000(1.0075)^{60} - \left\{\frac{a(r^n - 1)}{r-1}\right\}$$

$$= 30000(1.0075)^{60} - \left\{\frac{M(1.0075^{60} - 1)}{0.0075}\right\}$$

But $A_{60} = 0$

$$\therefore 30000(1.0075)^{60} - \left\{\frac{M(1.0075^{60} - 1)}{0.0075}\right\} = 0$$

$$M\left\{\frac{(1.0075^{60} - 1)}{0.0075}\right\} = 30000(1.0075)^{60}$$

$$M = \frac{30000(1.0075)^{60}(0.0075)}{(1.0075^{60} - 1)}$$

$$\therefore \underline{M = \$622.75}$$

(iii) *Finding the length of the loan*

2005 HSC Question 8c)

Weelabarrabak Shire Council borrowed \$3000000 at the beginning of 2005. The annual interest rate is 12%. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of \$480000, with the first repayment being made at the end of 2005.

Let A_n be the balance owing after the nth repayment.

(i) Show that $A_2 = \left(3\times10^6\right)(1.12)^2 - \left(4.8\times10^5\right)(1+1.12)$

Initial loan is borrowed for 2 years $= 3000000(1.12)^2$

1st repayment invested for 1 year $= 480000(1.12)^1$

2nd repayment invested for 0 years $= 480000$

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$

$$A_2 = (3000000)(1.12)^2 - \{480000 + 480000(1.12)\}$$

$$A_2 = \left(3\times10^6\right)(1.12)^2 - \left(4.8\times10^5\right)(1+1.12)$$

(ii) Show that $A_n = 10^6\left\{4-(1.12)^n\right\}$

Initial loan is borrowed for n years $= 3000000(1.12)^n$

1^{st} repayment invested for $n-1$ years $= 480000(1.12)^{n-1}$

2^{nd} repayment invested for $n-2$ years $= 480000(1.12)^{n-2}$

$\vdots$

2^{nd} last repayment invested for 1 year $= 480000(1.12)^1$

last repayment invested for 0 years $= 480000$

$$A_n = (\text{principal plus interest}) - (\text{instalments plus interest})$$

$$A_n = (3000000)(1.12)^n - (480000)\left\{1+1.12+\ldots+(1.12)^{n-2}+(1.12)^{n-1}\right\}$$

$$a = 480000, r = 1.12, n = n$$

$$= 3000000(1.12)^n - \left\{\frac{a\left(r^n-1\right)}{r-1}\right\}$$

$$A_n = 3000000(1.12)^n - \left\{\frac{480000\left(1.12^n - 1\right)}{0.12}\right\}$$

$$= 3000000(1.12)^n - 4000000\left(1.12^n - 1\right)$$

$$= 3000000(1.12)^n - 4000000(1.12)^n + 4000000$$

$$= 4000000 - 1000000(1.12)^n$$

$$= 10^6\left\{4 - (1.12)^n\right\}$$

(iii) In which year will Weelabarrabak Shire Council make the final repayment?

$$A_n = 0$$

$$10^6\left\{4 - (1.12)^n\right\} = 0$$

$$4 - (1.12)^n = 0$$

$$(1.12)^n = 4$$

$$\log(1.12)^n = \log 4$$

$$n \log 1.12 = \log 4$$

$$n = \frac{\log 4}{\log 1.12}$$

$$n = 12.2325075$$

The thirteenth repayment is the final repayment which will occur at the end of 2017

(iv) 2020 Mathematics HSC Question 26

Tina inherits \$60 000 and invests it in an account earning interest at a rate of 0.5% per month.
Each month, immediately after the interest has been paid, Tina withdraws \$800

The amount in the account immediately after the *n*th withdrawal can be determined using the recurrence relation

$$A_n = A_{n-1}(1.005) - 800$$

where $n = 1, 2, 3, \ldots$ and $A_0 = 60000$

a) Use the recurrence relation to find the amount of money in the account immediately after the third withdrawal.

$$\begin{aligned} A_3 &= A_2(1.005) - 800 \\ &= \left[A_1(1.005) - 800\right](1.005) - 800 \\ &= \left(\left[A_0(1.005) - 800\right](1.005) - 800\right)(1.005) - 800 \\ &= A_0(1.005)^3 - 800(1.005^2 + 1.005 + 1) \end{aligned}$$

$$A_3 = A_0(1.005)^3 - 800(1.005^2 + 1.005 + 1)$$
$$= 60000(1.005)^3 - 800(1.005^2 + 1.005 + 1)$$
$$= 58492.4875$$
$$= \$58492.49$$

b) Calculate the amount of interest earned in the first three months.

$$\text{Interest} = \underbrace{\$58492.49}_{\text{balance}} + 3 \times \underbrace{800}_{\text{plus withdrawals}} - \underbrace{60000}_{\text{minus principal}}$$
$$= \$892.49$$

c) Calculate the amount of money in the account immediately after the 94th withdrawal.

$$A_{94} = 60000(1.005)^{94} - 800(1.005^{93} + 1.005^{92} + \ldots + 1.005 + 1)$$
$$= 60000(1.005)^{94} - 800 \times \frac{1.005^{94} - 1}{1.005 - 1}$$
$$= 187.8459978\ldots$$
$$= \$187.85$$

Exercise 14E; 4, 8, 9, 13, 15