Sum and Product of Roots

If α and β are the roots of $ax^2+bx+c=0$, then;

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

$$ax^2+bx+c=a\left(x^2-\alpha x-\beta x+\alpha\beta\right)$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=x^2-(\alpha+\beta)x+\alpha\beta$$

Thus

$$\alpha+\beta=\frac{-b}{a} \qquad \text{(sum of roots)}$$

$$\alpha\beta=\frac{c}{a} \qquad \text{(product of roots)}$$

e.g. (i) Form a quadratic equation whose roots are;

$a)$ 2 and -3

$$\alpha + \beta = -1$$

$$\alpha\beta = -6$$

$$\underline{x^2 + x - 6 = 0}$$

$b)$ $2+\sqrt{5}$ and $2-\sqrt{5}$

$$\alpha + \beta = 4$$

$$\alpha\beta = 4 - 5$$

$$= -1$$

$$\underline{x^2 - 4x - 1 = 0}$$

(ii) If α and β are the roots of $2x^2 - 3x - 1 = 0$, find;

$a)$ $\alpha + \beta = \frac{-b}{a}$

$$= \underline{\frac{3}{2}}$$

$b)$ $\alpha\beta = \frac{c}{a}$

$$= \underline{\frac{-1}{2}}$$

$$c)\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$

$$=\left(\frac{3}{2}\right)^2-2\left(\frac{-1}{2}\right)$$

$$=\frac{9}{4}+1$$

$$=\frac{13}{4}$$

$$d)\ \frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

$$=\frac{\frac{3}{2}}{\frac{-1}{2}}$$

$$=-3$$

(iii) Find the value of m if one root is double the other in $x^2+6x+m=0$

Let the roots be α and 2α

$$\alpha+2\alpha=-6$$

$$3\alpha=-6$$

$$\alpha=-2$$

$$(\alpha)(2\alpha)=m$$

$$2\alpha^2=m$$

$$2(-2)^2=m$$

$$m=8$$

Roots and Coefficients

Quadratics $ax^2 + bx + c = 0$

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Cubics $ax^3 + bx^2 + cx + d = 0$

$$\alpha + \beta + \gamma = -\frac{b}{a} \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Quartics $ax^4 + bx^3 + cx^2 + dx + e = 0$

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \qquad \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a} \qquad \alpha\beta\gamma\delta = \frac{e}{a}$$

For the polynomial equation;

$$ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots = 0$$

$$\sum \alpha = -\frac{b}{a}$$ (sum of roots, one at a time)

$$\sum \alpha\beta = \frac{c}{a}$$ (sum of roots, two at a time)

$$\sum \alpha\beta\gamma = -\frac{d}{a}$$ (sum of roots, three at a time)

$$\sum \alpha\beta\gamma\delta = \frac{e}{a}$$ (sum of roots, four at a time)

Note:

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$$

e.g. (i) If α, β and γ are the roots of $2x^3 - 5x^2 - 3x + 1 = 0,$ find the values of;

a) $4\alpha + 4\beta + 4\gamma - 7\alpha\beta\gamma$

$$\alpha + \beta + \gamma = \frac{5}{2} \qquad \alpha\beta + \alpha\gamma + \beta\gamma = -\frac{3}{2} \qquad \alpha\beta\gamma = -\frac{1}{2}$$

$$4\alpha + 4\beta + 4\gamma - 7\alpha\beta\gamma = 4\left(\frac{5}{2}\right) - 7\left(-\frac{1}{2}\right)$$

$$= \frac{27}{2}$$

b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$$= \frac{-\frac{3}{2}}{-\frac{1}{2}}$$

$$= -3$$

c) $\alpha^2 + \beta^2 + \gamma^2$

$$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \left(\frac{5}{2}\right)^2 - 2\left(-\frac{3}{2}\right)$$

$$= \frac{37}{4}$$

1988 Extension 1 HSC Q2c)

If α, β and γ are the roots of $x^3 - 3x + 1 = 0$ find:

(i) $\alpha + \beta + \gamma$

$$\alpha + \beta + \gamma = 0$$

(ii) $\alpha\beta\gamma$

$$\alpha\beta\gamma = -1$$

(iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}$$

$$= \frac{-3}{-1}$$

$$= 3$$

2003 Extension 1 HSC Q4c)

It is known that two of the roots of the equation $2x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other.
Find the value of k.

Let the roots be $\alpha, \frac{1}{\alpha}$ and β

$$(\alpha)\left(\frac{1}{\alpha}\right)(\beta) = \frac{-6}{2}$$

$$\beta = -3$$

$$P(-3) = 0$$

$$2(-3)^3 + (-3)^2 - k(-3) + 6 = 0$$

$$-54 + 9 + 3k + 6 = 0$$

$$3k = 39$$

$$\underline{k = 13}$$

2006 Extension 1 HSC Q4a)

The cubic polynomial $P(x) = x^3 + rx^2 + sx + t$, where r, s and t are real numbers, has three real zeros, 1, α and $-\alpha$

(i) Find the value of r

$$1 + \alpha + -\alpha = -r$$

$$r = -1$$

(ii) Find the value of $s + t$

$$(1)(\alpha) + (1)(-\alpha) + (\alpha)(-\alpha) = s$$

$$s = -\alpha^2$$

$$(1)(\alpha)(-\alpha) = -t$$

$$t = \alpha^2$$

$$\therefore s + t = 0$$

OR

$$P(1) = 0$$

$$1 + r + s + t = 0$$

$$1 - 1 + s + t = 0$$

$$\therefore s + t = 0$$

Exercise 10F; 1, 2, 3, 4, 6, 7ac, 8a, 9a, 10ad, 11, 13ad, 14, 16, 17, 18, 19