The Logic of an Induction Proof

An induction proof is a series of statements linked using logical connectives

***S*(*n*)**: some statement that needs to be proved for all cases

***S*(1)**: the statement is true for the first case *(base case)*

***S*(*k*)**: the statement is true for a generalised case *(assumption)*

***S*(*k* + 1)**: the statement is true for next case after the generalised case

$$\boxed{\{S(1) \wedge (S(k) \Rightarrow S(k+1)\} \Rightarrow S(n)}$$

and

implies
(if then)

The Structure of an Induction Proof

Step 1: Prove the result is true for $n = 1$ (or whatever the first term is)

Step 2: Assume the result is true for $n = k$, where k is a positive integer (or another condition that matches the question)

or using set notation;
Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$

Step 3: Prove the result is true for $n = k + 1$

NOTE: It is important to note in your conclusion that the result is true for $n = k + 1$ **if it is true for $n = k$**

Step 4: Since the result is true for $n = 1$, then the result is true for all positive integral values of n by induction

or using set notation;
Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

$$e.g.(i)\ 1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

Prove the result is true for $n = 1$

$$LHS = 1^2 \qquad\qquad RHS = \frac{1}{3}(1)(2-1)(2+1)$$

$$= 1 \qquad\qquad = \frac{1}{3}(1)(1)(3)$$

$$= 1$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 1$ $\boxed{\boldsymbol{S(1)}}$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$i.e.\ 1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$$

Prove the result is true for $n = k + 1$

$$i.e. \text{ Prove: } 1^2+3^2+5^2+\ldots+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$$

Proof:

$$1^2+3^2+5^2+\ldots+(2k+1)^2$$

$$=1^2\underbrace{+3^2+5^2+\ldots+(2k-1)^2}_{S_k}+\underbrace{(2k+1)^2}_{T_{k+1}}$$

$$=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2 \qquad \text{(by assumption)}$$

$$=(2k+1)\left[\frac{1}{3}k(2k-1)+(2k+1)\right]$$

$$=\frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$$

$$=\frac{1}{3}(2k+1)\left(2k^2-k+6k+3\right)$$

$$=\frac{1}{3}(2k+1)\left(2k^2+5k+3\right)$$

$$=\frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$\boxed{\boldsymbol{S(k) \Rightarrow S(k+1)}}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\ n \in \mathbb{Z}^+$ by induction $\boxed{\boldsymbol{S(n)}}$

$(ii)\ \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Prove the result is true for $n = 1$

$$LHS = \frac{1}{1\times 3} \qquad RHS = \frac{1}{2+1}$$

$$= \frac{1}{3} \qquad = \frac{1}{3}$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$i.e. \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Prove the result is true for $n = k + 1$

$$i.e.\ \text{Prove}: \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Proof:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \qquad \text{(by assumption)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)}{(2k+3)}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\ n \in \mathbb{Z}^+$ by induction

The Three Key Parts of an Induction Proof

The setup

1. prove true for first case

2. assume what it is that you are asked to prove

3. state what you are going to try to prove

The proof

it is a deductive proof so;

- *provide explanations for "non-obvious" steps*
- *conclude with an if – then statement*

tie the two parts together with your conclusion

Exercise 2A; 2acfh, 3, 4b, 5c, 6, 7, 8, 9, 11ac, 12, 13