nth Roots Of Unity $\left(z^n = \pm 1\right)$

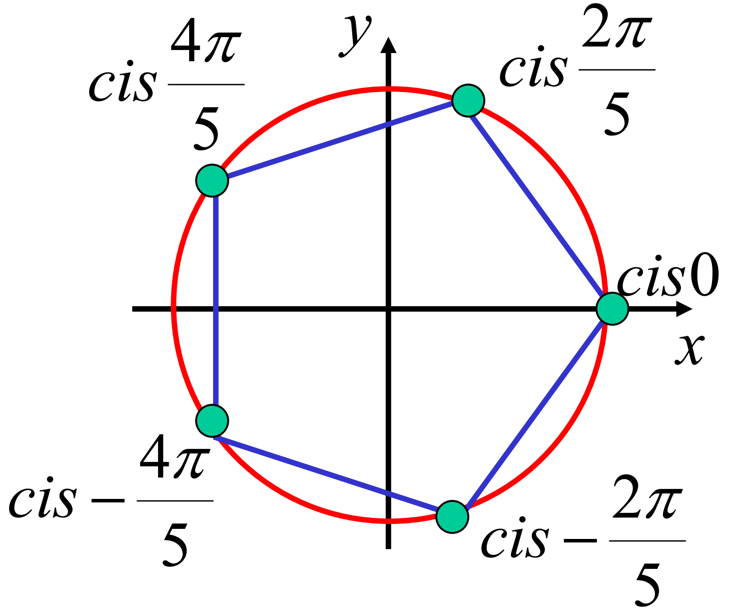
The solutions of equations of the form, $z^n = \pm 1$, are the nth roots of unity

When placed on an Argand Diagram, they form a regular n sided polygon, with vertices on the unit circle.

$e.g.\ z^5 = 1$

$$z = cis\left[\frac{2\pi k + 0}{5}\right] \quad k = 0, \pm 1, \pm 2$$

$$z = cis0, cis\frac{2\pi}{5}, cis-\frac{2\pi}{5}, cis\frac{4\pi}{5}, cis-\frac{4\pi}{5}$$



b) (i) If ω is a complex root of $z^5 - 1 = 0$, show that $\omega^2, \omega^3, \omega^4$ and ω^5 are the other roots.

A complex root means a non-real root

$z^5 = 1$ If ω is a solution then $\omega^5 = 1$

$$\therefore (\omega^5)^5 = 1^5$$

$$= 1 \quad \therefore \omega^5 \text{ is a solution}$$

$$(\omega^2)^5 = (\omega^5)^2 = 1^2 = 1$$

$\therefore \omega^2$ is a solution

$$(\omega^3)^5 = (\omega^5)^3 = 1^3 = 1$$

$\therefore \omega^3$ is a solution

$$(\omega^4)^5 = (\omega^5)^4 = 1^4 = 1$$

$\therefore \omega^4$ is a solution

Thus if ω is a root then $\omega^2, \omega^3, \omega^4$ and ω^5 are also roots

OR $z^5 = 1$ If ω is a solution then $\omega^5 = 1$

If $k \in \mathbb{Z}$ then $(\omega^k)^5 = (\omega^5)^k$

$$= 1^k = 1$$

Thus if ω is a root then $\omega^2, \omega^3, \omega^4$ and ω^5 are also roots

(ii) Prove that $1+\omega+\omega^2+\omega^3+\omega^4=0$

$$\omega+\omega^2+\omega^3+\omega^4+\omega^5=-\frac{b}{a} \qquad \text{(sum of the roots)}$$

$$1+\omega+\omega^2+\omega^3+\omega^4=0 \qquad (\omega^5=1)$$

OR $\omega^5-1=0$

$$(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4)=0$$

$$\therefore 1+\omega+\omega^2+\omega^3+\omega^4=0 \qquad (\omega\neq 1)$$

$$\left[\begin{array}{l} NOTE: \\ \omega^n-1=0 \\ (\omega-1)(1+\omega+\omega^2+\ldots+\omega^{n-1})=0 \end{array}\right]$$

OR

$$1+\omega+\omega^2+\omega^3+\omega^4=\frac{a(r^n-1)}{r-1} \qquad \text{GP: } a=1, r=\omega, n=5$$

$$=\frac{1(\omega^5-1)}{\omega-1} \quad =0 \qquad (\omega^5-1=0)$$

(iii) Find the quadratic equation whose roots are $\omega+\omega^4$ and $\omega^2+\omega^3$

$$\alpha+\beta=\omega+\omega^4+\omega^2+\omega^3$$
$$=-1$$

$$\alpha\beta=(\omega+\omega^4)(\omega^2+\omega^3)$$
$$=\omega^3+\omega^4+\omega^6+\omega^7$$
$$=\omega^3+\omega^4+\omega+\omega^2$$
$$=-1$$

$\therefore$ equation is $x^2+x-1=0$

c) If ω is a complex cube root of unity, use the fact that $1+\omega+\omega^2=0$ to;

(i) Evaluate $\left(1+\omega^2\right)^3$

$$=(-\omega)^3$$

$$=-\omega^3$$

$$=-1$$

(ii) Evaluate $\dfrac{1}{1+\omega}+\dfrac{1}{1+\omega^2}$

$$=-\frac{1}{\omega^2}-\frac{1}{\omega}$$

$$=\frac{-1-\omega}{\omega^2}$$

$$=\frac{\omega^2}{\omega^2}$$

$$=1$$

(iii) Form the cubic equation with roots $1,\ 1+\omega,\ 1+\omega^2$

$$(z-1)\left\{z^2-\left(2+\omega+\omega^2\right)z+\left(1+\omega+\omega^2+\omega^3\right)\right\}=0$$

$$(z-1)\left(z^2-z+1\right)=0$$

$$z^3-z^2+z-z^2+z-1=0$$

$$z^3-2z^2+2z-1=0$$

d) Solve $z^5 + z^4 + z^3 + z^2 + z + 1 \;= 0$

Now $z^6 - 1 = (z-1)\left(z^5 + z^4 + z^3 + z^2 + z + 1\right)$

And $z^6 - 1 = 0$ has solutions;

$$z = cis\left[\frac{2\pi k}{6}\right] \; k = 0, \pm 1, \pm 2, 3$$

$$z^5 + z^4 + z^3 + z^2 + 1 = 0$$

$$\frac{(z-1)\left(z^5 + z^4 + z^3 + z^2 + 1\right)}{(z-1)} = 0$$

$$z^6 - 1 = 0, z \neq 1$$

$$z = cis\frac{\pi}{3}, cis - \frac{\pi}{3}, cis\frac{2\pi}{3}, cis - \frac{2\pi}{3}, cis\pi$$

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -1$$

e) 1996 HSC

Let $\omega = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$

(i) Show that ω^k is a solution of $z^9 - 1 = 0$, where k is an integer

$$z^9 = 1$$

$$z = cis\left[\frac{2\pi k}{9}\right] \qquad k = 0,1,2,3,4,5,6,7,8$$

$$z = \left\{cis\frac{2\pi}{9}\right\}^k$$

$$\underline{z = \omega^k}$$

(ii) Prove that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$

$$z^9 - 1 = 0$$

$$\therefore 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0 \qquad \text{(sum of roots)}$$

$$\underline{\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1}$$

(iii) Hence show that $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$

$$z^9-1$$

$$=(z-1)(z-\omega)(z-\omega^8)(z-\omega^2)(z-\omega^7)(z-\omega^3)(z-\omega^6)(z-\omega^4)(z-\omega^5)$$

$$=(z-1)\left(z^2-2\cos\frac{2\pi}{9}z+1\right)\left(z^2-2\cos\frac{4\pi}{9}z+1\right)$$

$$\left(z^2-2\cos\frac{6\pi}{9}z+1\right)\left(z^2-2\cos\frac{8\pi}{9}z+1\right)$$

$$=(z-1)\left(z^2-2\cos\frac{2\pi}{9}z+1\right)\left(z^2-2\cos\frac{4\pi}{9}z+1\right)$$

$$\left(z^2+z+1\right)\left(z^2-2\cos\frac{8\pi}{9}z+1\right)$$

Let $z=i$

$$i^9-1=(i-1)\left(-2\cos\frac{2\pi}{9}i\right)\left(-2\cos\frac{4\pi}{9}i\right)(i)\left(-2\cos\frac{8\pi}{9}i\right)$$

$$i^9 - 1 = (i-1)\left(-2\cos\frac{2\pi}{9} i\right)\left(-2\cos\frac{4\pi}{9} i\right)(i)\left(-2\cos\frac{8\pi}{9} i\right)$$

$$i - 1 = -i^4 (i-1)\left(2\cos\frac{2\pi}{9}\right)\left(2\cos\frac{4\pi}{9}\right)\left(2\cos\frac{8\pi}{9}\right)$$

$$-1 = 8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}$$

$$-1 = 8\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\cos\frac{\pi}{9}\right)$$

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$$

Exercise 3C; 1 to 4, 5ac, 6, 7, 8, 9, 11, 12, 13