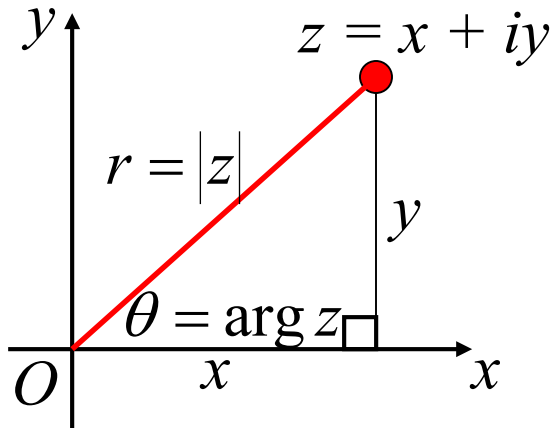


# *Mod-Arg Form*

## **Modulus**

The modulus of a complex number is the length of the vector  $OZ$



$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$|z| = \sqrt{x^2 + y^2}$$

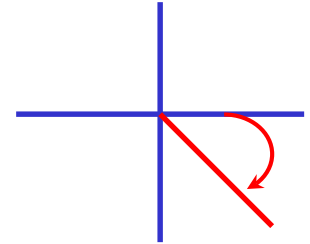
## **Argument**

The argument of a complex number is the angle the vector  $OZ$  makes with the positive real ( $x$ ) axis

$$\arg z = \tan^{-1}\left(\frac{y}{x}\right) \quad -\pi < \arg z \leq \pi$$

e.g. Find the modulus and argument of  $4 - 4i$

$$\begin{aligned} |4 - 4i| &= \sqrt{4^2 + (-4)^2} & \arg(4 - 4i) &= \tan^{-1}\left(\frac{-4}{4}\right) \\ &= \sqrt{32} & &= \tan^{-1}(-1) \\ &= \underline{4\sqrt{2}} & &= \underline{-\frac{\pi}{4}} \end{aligned}$$



Every complex number can be written in terms of its modulus and argument

$$\begin{aligned} z &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

The **mod-arg** form of  $z$  is;

$$z = r(\cos \theta + i \sin \theta)$$

$$z = r \operatorname{cis} \theta$$

where;  $r = |z|$

$$\theta = \arg z$$

e.g. (i)  $4 - 4i = \underline{4\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)}$

(ii)  $\sqrt{3} + i$

$$\begin{aligned}|z| &= \sqrt{(\sqrt{3})^2 + 1^2} \\ &= \sqrt{4} \\ &= 2\end{aligned}$$

$$\begin{aligned}\arg z &= \tan^{-1} \frac{1}{\sqrt{3}} \\ &= \frac{\pi}{6}\end{aligned}$$

$$\therefore \underline{\sqrt{3} + i = 2\text{cis}\frac{\pi}{6}}$$

(ii) Convert  $6\text{cis}\frac{\pi}{6}$  to Cartesian form

$$\begin{aligned}6\text{cis}\frac{\pi}{6} &= 6\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= 6\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= \underline{3\sqrt{3} + 3i}\end{aligned}$$

# *Mod-Arg Relations*

$$(1) |z_1 z_2| = |z_1| |z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

**NOTE:**

Multiplication rotates  $z_1$  by  $\arg z_2$

**Proof:** let  $z_1 = r_1 \operatorname{cis} \theta_1$  and  $z_2 = r_2 \operatorname{cis} \theta_2$

$$z_1 z_2 = r_1 (\cos \theta_1 + i \sin \theta_1) \times r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 r_2 (\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 - \sin \theta_1 \sin \theta_2)$$

$$= r_1 r_2 \{ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \}$$

$$= r_1 r_2 \{ \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \}$$

$$\therefore |z_1 z_2| = r_1 r_2$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2$$

$$= \underline{|z_1| |z_2|}$$

$$= \underline{\arg z_1 + \arg z_2}$$

**NOTE:** it follows that:

$$|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$

$$\arg(z_1 z_2 z_3 \dots z_n) = \arg z_1 + \arg z_2 + \arg z_3 + \dots + \arg z_n$$

$$(2) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

**NOTE:** it follows that;

$$\left| \frac{z_1 z_2}{z_3 z_4} \right| = \frac{|z_1| |z_2|}{|z_3| |z_4|}$$

$$\arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = \arg z_1 + \arg z_2 - \arg z_3 - \arg z_4$$

$$(3) |z^n| = |z|^n$$

$$\arg(z^n) = n \arg z$$

e.g. Find the modulus and argument of  $z = \frac{(5+i)(-2-i)}{3+2i}$

$$\begin{aligned}|z| &= \frac{\sqrt{5^2 + 1^2} \sqrt{(-2)^2 + (-1)^2}}{\sqrt{3^2 + 2^2}} \\ &= \frac{\sqrt{26} \sqrt{5}}{\sqrt{13}} \\ &= \underline{\underline{\sqrt{10}}}\end{aligned}$$

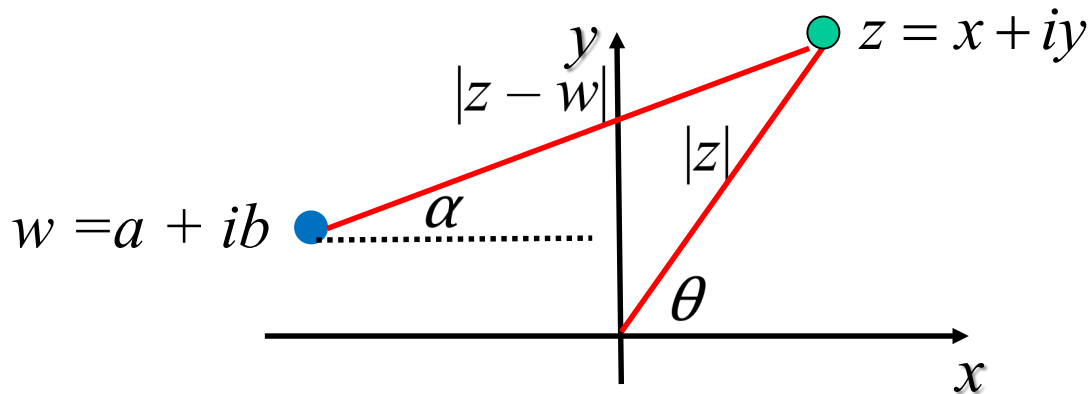
$$\begin{aligned}\arg z &= \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{-1}{-2}\right) - \tan^{-1}\left(\frac{2}{3}\right) \\ &= 11^\circ 19' + (-153^\circ 26') - 33^\circ 41' \\ &= \underline{\underline{-175^\circ 48'}}\end{aligned}$$

# *Shifting the Reference Point*

In the Cartesian plane, the reference point is sometimes moved from the origin to another point

e.g.  $(x - 1)^2 + (y - 2)^2 = 1$ , the centre of the circle has moved from the origin to  $(1,2)$

A similar process is used in the Argand Diagram



$|z|$  = distance from  $z$  to the origin

$\arg z$  = direction of the position vector  $\vec{OZ}$  ( $\theta$ )

$|z - w|$  = distance from  $z$  to  $w$

$\arg(z - w)$  = direction of the displacement vector  $\vec{WZ}$  ( $\alpha$ )

**Exercise 1D; 1ace, 2acf, 3bdf, 4ac,  
5bcf, 6, 7adf, 8ace, 9ace, 13, 15,  
16, 17, 18, 20, 21, 22, 24**