

Applications of Euler's Formula

Exponential Functions

$$\begin{aligned}e^{i\theta + 2ik\pi} &= \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) \\ &= \cos\theta + i\sin\theta \text{ if } k \in \mathbb{Z} \\ &= e^{i\theta}\end{aligned}$$

$$e^{i(\theta + 2\pi k)} = e^{i\theta}$$

So e^z is a periodic function, with period $2\pi i$

Trigonometric Functions

Earlier we proved the following identities:

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

rearranging these we get;

$$2\cos n\theta = e^{n\theta i} + e^{-n\theta i} \quad 2i\sin n\theta = e^{n\theta i} - e^{-n\theta i}$$
$$\cos n\theta = \frac{1}{2}(e^{n\theta i} + e^{-n\theta i}) \quad \sin n\theta = \frac{1}{2i}(e^{n\theta i} - e^{-n\theta i})$$

e.g. Prove the identities;

(i) $\sin^2 \theta + \cos^2 \theta = 1$

$$\begin{aligned}\sin^2 \theta + \cos^2 \theta &= \frac{1}{(2i)^2} (e^{i\theta} - e^{-i\theta})^2 + \frac{1}{2^2} (e^{i\theta} + e^{-i\theta})^2 \\ &= \frac{1}{4} (-e^{2i\theta} + 2 - e^{-2i\theta} + e^{2i\theta} + 2 + e^{-2i\theta}) \\ &= \frac{1}{4}(4) = \underline{1}\end{aligned}$$

$$(ii) \sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3 \theta$$

$$\begin{aligned} \sin^3 \theta &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^3 \\ &= \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{-8i} \\ &= \frac{1}{4} \left(\frac{3(e^{i\theta} - e^{-i\theta}) - (e^{3i\theta} - e^{-3i\theta})}{2i} \right) \\ &= \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3 \theta \end{aligned}$$

$$(iii) 2\sin\alpha\cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

$$\begin{aligned} 2\sin\alpha\cos\beta &= 2 \times \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \times \frac{e^{i\beta} + e^{-i\beta}}{2} \\ &= \frac{e^{i(\alpha + \beta)} + e^{i(\alpha - \beta)} - e^{-i(\alpha - \beta)} - e^{-i(\alpha + \beta)}}{2i} \\ &= \frac{e^{i(\alpha + \beta)} - e^{-i(\alpha + \beta)}}{2i} + \frac{e^{i(\alpha - \beta)} - e^{-i(\alpha - \beta)}}{2i} \\ &= \underline{\sin(\alpha + \beta) + \sin(\alpha - \beta)} \end{aligned}$$

Roots of Complex Numbers

e.g. Find the square roots of $1 - \sqrt{3} i$

$$\begin{aligned} z^2 &= 1 - \sqrt{3} i \\ &= 2e^{-\frac{i\pi}{3}} \end{aligned}$$

$$e^z = e^{z + 2i\pi k}$$

$$z = \sqrt{2} e^{-\frac{i\pi}{6} + \pi k}, k = 0, 1$$

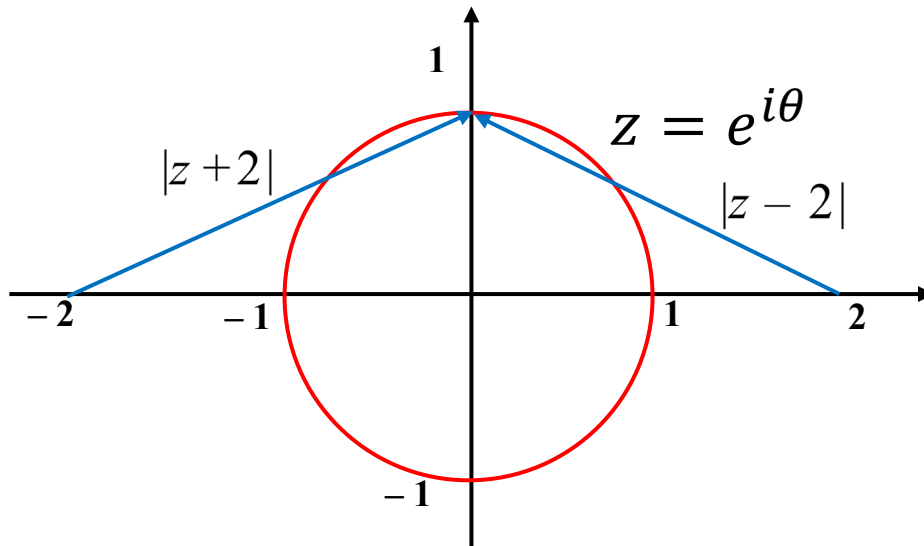
$$z = \sqrt{2} e^{-\frac{i\pi}{6}}, \sqrt{2} e^{\frac{5i\pi}{6}}$$

$$\begin{aligned} z &= \sqrt{2} \left(\frac{\sqrt{3}}{2} - \frac{1}{2} i \right), \sqrt{2} \left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i \right) \\ &= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i, -\frac{\sqrt{6}}{2} + \frac{1}{2} i \end{aligned}$$

e.g. 2020 Extension 2 HSC Q9

What is the maximum value of $|e^{i\theta} - 2| + |e^{i\theta} + 2|$ for $0 \leq \theta \leq 2\pi$?

$$\text{Let } z = e^{i\theta} \Rightarrow |z| = 1$$



$|z - 2|$ is the length of the vector joining z to 2

$|z + 2|$ is the length of the vector joining z to -2

maximum $|z - 2| + |z + 2|$ is when $z = \pm i$

$$\begin{aligned} |i - 2| &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5} \end{aligned}$$

$$\therefore \max |z - 2| + |z + 2| = 2\sqrt{5}$$

**Exercise 3E; 1, 3,
4bc, 6, 7, 8b ii, iii,
11bc, 14**