

# *Deductive Reasoning*

An argument is valid iff it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless false.

## (1) Direct Proof (*modus ponens*)

$$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$$

$P$	$Q$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$						
T	T	T	T	T	T	T	T	T
T	F	T	F	T	F	F	T	F
F	T	F	F	F	T	T	T	T
F	F	F	F	F	T	F	T	F

$$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

e.g. (i) Prove that if a number is odd, then its square is also odd

Let  $n$  be an odd integer

$$n = 2k + 1, \quad k \in \mathbb{Z}$$

$$n^2 = (2k + 1)^2$$

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2P + 1 \quad \text{where } P = (2k^2 + 2k) \in \mathbb{Z}$$

hence if  $n$  is odd,  $n^2$  is also odd

**(ii) 2021 Extension 2 HSC Question 15 d)**

Prove that  $2^n + 3^n \neq 5^n$  for all integers  $n \geq 2$

$$5^n = (2 + 3)^n$$

$$= \binom{n}{0} 2^n + \binom{n}{1} 2^{n-1} 3 + \binom{n}{2} 2^{n-2} 3^2 + \dots + \binom{n}{n} 3^n$$

$$> 2^n + 3^n \quad \forall n \in \mathbb{Z} : n \geq 2$$

thus  $5^n \neq 2^n + 3^n$  for  $n \geq 2$

(iii) Prove that the sum of the squares of five consecutive integers is divisible by 5

If  $p, q \in \mathbb{Z}$  and  $q$  is divisible by  $p$  then  $\exists n \in \mathbb{Z} : q = pn$

$$\begin{aligned}\text{Let } q &= (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2, \quad n \in \mathbb{Z} \\ &= 5n^2 + 4 + 1 + 1 + 4 \\ &= 5n^2 + 10 \\ &= 5(n^2 + 2) \\ &= 5P \quad \text{where } P = (n^2 + 2) \in \mathbb{Z}\end{aligned}$$

hence the sum of the squares of five consecutive integers is divisible by 5

## (2) Proof by Contraposition (*modus tollens*)

$$(\neg Q \Rightarrow \neg P) \Leftrightarrow (P \Rightarrow Q)$$

e.g. Prove that if  $2^n - 1$ ,  $n \in \mathbb{N}$ , is prime then  $n$  is prime

Let  $n = pq$ ,  $p, q \in \mathbb{N}$  and  $p, q \neq 1$  (i.e.  $n$  is not prime)

$$2^n - 1 = 2^{pq} - 1$$

$$= (2^p)^q - 1$$

$$= (2^p - 1) \left( 1 + 2^p + 2^{2p} + \dots + 2^{(q-1)p} \right)$$

$$= PQ, \text{ where } P = (2^p - 1) \neq 1$$

$$\text{and } Q = \left( 1 + 2^p + 2^{2p} + \dots + 2^{(q-1)p} \right) \neq 1 \forall P, Q \in \mathbb{N}$$

$\therefore$  if  $n$  is not prime, then  $2^n - 1$  is not prime

hence if  $2^n - 1$  is prime then  $n$  is prime, by contraposition

### (3) Proof by Contradiction (*reductio ad impossibile* – indirect proof)

$$(\neg(P \Rightarrow Q) \Rightarrow (R \wedge \neg R)) \Rightarrow (P \Rightarrow Q)$$

$P$	$Q$	$(\neg(P \Rightarrow Q) \Rightarrow (R \wedge \neg R)) \Rightarrow (P \Rightarrow Q)$							
T	T	F	T	F	T	T	T	T	
T	F	T	F	F	T	T	F	F	
F	T	F	T	F	T	F	T	T	
F	F	F	T	F	T	F	T	F	

e.g. (i) Prove  $\log_2 5$  is irrational

Assume  $\log_2 5$  is rational

$$\text{i.e. } \log_2 5 = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are coprime}$$

$$2^q = 5$$

$$2^p = 5^q$$

so LHS is even and RHS is odd, which is a contradiction

$\therefore \log_2 5$  is irrational

(ii) Prove that there are no integers  $a$  and  $b$  such that  $18a + 6b = 1$

Assume  $\exists a, b \in \mathbb{Z} : 18a + 6b = 1$

$$18a + 6b = 1$$

$$6(3a + b) = 1$$

$$3a + b = \frac{1}{6}$$

however  $3a + b \in \mathbb{Z}$

$3a + b \neq \frac{1}{6}$ , which is a contradiction

Thus there are no integers  $a$  and  $b$  such that  $18a + 6b = 1$

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## e.g. 2020 Extension 2 HSC Question 15

In the set of integers, let  $P$  be the proposition:

“If  $k + 1$  is divisible by 3, then  $k^3 + 1$  is divisible by 3”

(i) Prove that the proposition is true

Let  $k + 1 = 3P$  , where  $P \in \mathbb{Z} \forall k \in \mathbb{Z}$

$$k^3 + 1 = (k + 1)(k^2 - k + 1)$$

$$= 3P(k^2 - k + 1)$$

$$= 3Q \quad \text{where } Q = P(k^2 - k + 1) \in \mathbb{Z}$$

Thus if  $k + 1$  is divisible by 3, then  $k^3 + 1$  is divisible by 3

(ii) Write down the contrapositive of the proposition  $P$

If  $k^3 + 1$  is not divisible by 3 then  $k + 1$  is not divisible by 3,

(iii) Write down the converse of the proposition  $P$  and state, with reasons, whether this converse is true or false

If  $k^3 + 1$  is divisible by 3 then  $k + 1$  is divisible by 3

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$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

$P$  :  $k^3 + 1$  is divisible by 3       $Q$  :  $k + 1$  is divisible by 3

$$\begin{aligned} k^3 + 1 &= (k + 1)(k^2 - k + 1) \\ &= (k + 1)(k^2 + 2k + 1 - 3k) \\ &= (k + 1)[(k + 1)^2 - 3k] \end{aligned}$$

If  $k + 1$  is not divisible by 3 then neither is  $(k + 1)^2$

Thus  $[(k + 1)^2 - 3k]$  is not divisible by 3, which means

$(k + 1)[(k + 1)^2 - 3k]$  is not divisible by 3

i.e. If  $k + 1$  is not divisible by 3 then  $k^3 + 1$  is not divisible by 3

$\therefore$  If  $k^3 + 1$  is divisible by 3 then  $k + 1$  is divisible by 3 by contraposition

**Exercise 2B;**  
**1b, 2c, 3a, 4a,**  
**7, 10, 12, 16,**  
**17b**

**Exercise 2C;**  
**1, 3, 4, 5,**  
**7, 9, 11, 13**