Integration Using Substitution

"NON" STANDARD INTEGRALS

Not listed on the reference sheet, however will save time if you know

$$\int \frac{du}{\sqrt{u}} = 2\sqrt{u}$$

$$\int \frac{du}{u^2} = -\frac{1}{u}$$

$$\int \ln x dx = x\ln x - x$$

$$\int \tan x dx = \log \sec x$$

* $$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

* $$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

* $$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

*** used to be on reference sheet might be given or asked to prove**

e.g. (*i*) $\int x\sqrt{x^2+4}\,dx$

$x \leftarrow \frac{1}{2}du$, $\quad x^2+4 \leftarrow u$, $\quad dx \leftarrow \frac{1}{2}du$

$$= \frac{1}{2}\int 2x\sqrt{x^2+4}\,dx$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}u^{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+4\right)^{\frac{3}{2}} + c$$

$$= \frac{1}{3}\left(x^2+4\right)\sqrt{x^2+4} + c$$

$$u = x^2 + 4$$

$$du = 2x\,dx$$

when substituting
***u* = *f*(*x*)**
make the function causing the problem *u*

OR

e.g. (i) $\int x\sqrt{x^2+4}dx = \int 2\tan\theta\sqrt{4\tan^2\theta+4}\cdot 2\sec^2\theta d\theta$

$$= 8\int \tan\theta\sec^3\theta$$

$$= 8\int \frac{\sin\theta d\theta}{\cos^4\theta}$$

$$= -8\int \frac{du}{u^4}$$

$$= -8\times -\frac{1}{3}u^{-3} + c$$

$$= \frac{8}{3}\sec^3\theta + c$$

$$= \frac{1}{3}\left(\sqrt{x^2+4}\right)^3 + c$$

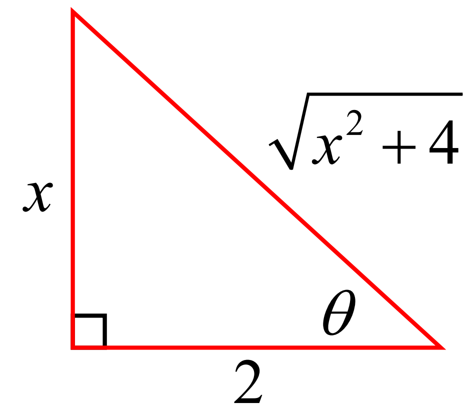
$$= \frac{1}{3}\left(x^2+4\right)\sqrt{x^2+4} + c$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta d\theta$

$u = \cos\theta$

$du = -\sin\theta d\theta$



if a root of a sum or difference of squares is involved, could try a trig substitution

Keep an eye out for
$f'(x)\times f(x)$

(ii) $\int x\sqrt{x+1}\,dx$ $\quad u=\sqrt{x+1} \Rightarrow x=u^2-1$

$dx = 2u\,du$

$$= \int (u^2-1)u \times 2u\,du$$

$$= 2\int (u^4-u^2)\,du$$

$$= \frac{2u^5}{5} - \frac{2u^3}{3} + c$$

$$= \frac{2(x+1)^{\frac{5}{2}}}{5} - \frac{2(x+1)^{\frac{3}{2}}}{3} + c$$

$$= \frac{2(x+1)^{\frac{3}{2}}}{15}[3(x+1)-5] + c$$

$$= \frac{2}{15}(3x-2)(x+1)\sqrt{x+1} + c$$

OR

(ii) $\int x\sqrt{x+1}\,dx$

$$= \int (x+1-1)\sqrt{x+1}\,dx$$

$$= \int \left\{ (x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}} \right\} dx$$

$$= \frac{2(x+1)^{\frac{5}{2}}}{5} - \frac{2(x+1)^{\frac{3}{2}}}{3} + c$$

$$= \frac{2(x+1)^{\frac{3}{2}}}{15}[3(x+1)-5] + c$$

$$= \frac{2}{15}(3x-2)(x+1)\sqrt{x+1} + c$$

(iii) $\int_1^4 \frac{dx}{(1+\sqrt{x})^2\sqrt{x}}$ $= 2\int_2^3 \frac{du}{u^2}$

$$= -2\left[\frac{1}{u}\right]_2^3$$

$$= -2\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{3}$$

$$u = 1 + \sqrt{x}$$

$$du = \frac{dx}{2\sqrt{x}}$$

when $x = 1$, $u = 2$

$x = 4$, $u = 3$

It is important to separate your variables, before substituting

- **NEVER substitute** $dx = 2\sqrt{x}du$
- **If your new integral is *du*, your limits MUST also be with respect to *u***

$$\text{(iv)} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x \cos x\, dx = \int_{\frac{\sqrt{3}}{2}}^{1} u^5\, du$$

$$u = \sin x$$

$$du = \cos x\, dx$$

$$x = \frac{\pi}{3}, u = \sin\frac{\pi}{3}$$

$$u = \frac{\sqrt{3}}{2}$$

$$= \frac{1}{6}\left[u^6\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$= \frac{1}{6}\left\{1^6 - \left(\frac{\sqrt{3}}{2}\right)^6\right\}$$

$$x = \frac{\pi}{2}, u = \sin\frac{\pi}{2}$$

$$u = 1$$

$$= \frac{37}{384}$$

$$\text{(v)} \int_{3}^{4} \frac{x\,dx}{\sqrt{25 - x^2}} = -\frac{1}{2}\int_{16}^{9} \frac{du}{\sqrt{u}}$$

$$u = 25 - x^2$$

$$du = -2x\,dx$$

$$= \left[\sqrt{u}\right]_{9}^{16}$$

$$x = 3, u = 16$$

$$x = 4, u = 9$$

$$= \sqrt{16} - \sqrt{9}$$

$$= 1$$

(vi) $\int_0^5 \sqrt{25 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{25 - 25\sin^2 u} \cdot 5\cos u\,du$ $\quad x = 5\sin u \Rightarrow u = \sin^{-1}\frac{x}{5}$

$dx = 5\cos u\,du$

$= \int_0^{\frac{\pi}{2}} \sqrt{25\cos^2 u} \cdot 5\cos u\,du$

$x = 0, u = \sin^{-1} 0$

$u = 0$

$= 25\int_0^{\frac{\pi}{2}} \cos^2 u\,du$

$x = 5, u = \sin^{-1} 1$

$u = \frac{\pi}{2}$

$= \frac{25}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2u)\,du$

$= \frac{25}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\frac{\pi}{2}}$

$= \frac{25}{2}\left\{\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0\right\}$

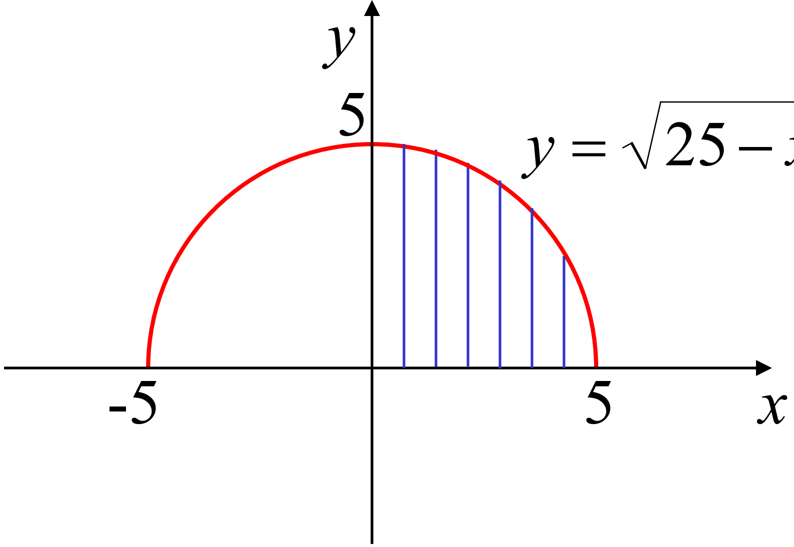
$= \frac{25\pi}{4}$

(vi) $\int_0^5 \sqrt{25 - x^2}\,dx$

$= \frac{1}{4}\pi(5)^2$

$= \frac{25\pi}{4}$

OR



Exercise 4C; 1ce, 2cde, 3cdf, 4, 5bc, 6ab, 7a, 8bd, 9, 10, 11, 12, 13, 14, 15

NOTE: substitution is not necessarily given in Extension 2