Integration Using Substitution

Try to convert the integral into a "standard integral"

STANDARD INTEGRALS on the reference sheet

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c \quad \text{where } n \neq -1$$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - [f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + [f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

du

e.g. (i) $\int \frac{x + 1}{4x^2 + 8x - 7}dx = \frac{1}{8}\int \frac{du}{u}$

u

Let $u = 4x^2 + 8x - 7$

$du = (8x + 8)dx$

$$= \frac{1}{8}\log|u| + c$$

$$= \frac{1}{8}\log|4x^2 + 8x - 7| + c$$

(ii) $$\int \frac{dx}{x(\log x)^3} = \int \frac{du}{u^3}$$

Let $u = \log x$

$$du = \frac{dx}{x}$$

$$= \int u^{-3} du$$

$$= -\frac{1}{2}u^{-2} + c$$

$$= -\frac{1}{2u^2} + c$$

$$= -\frac{1}{2(\log x)^2} + c$$

(iii) 2018 Extension 1 HSC Question 11f)

Evalute $\int_{-3}^{0} \frac{x}{\sqrt{1-x}} dx$, using the substitution $u = 1 - x$

$$\int_{-3}^{0} \frac{x\,dx}{\sqrt{1-x}} = -\int_{4}^{1} \frac{1-u}{\sqrt{u}} du$$

$$= \int_{1}^{4} \left(\frac{1}{\sqrt{u}} - \sqrt{u} \right) du$$

$$= \left[2\sqrt{u} - \frac{2}{3} u \sqrt{u} \right]_1^4$$

$$= 4 - \frac{16}{3} - 2 + \frac{2}{3}$$

$$= -\frac{8}{3}$$

$u = 1 - x$

$du = -dx$

when $x = -3$, $u = 4$

$x = 0$, $u = 1$

(iv) 2019 Extension 1 HSC Question 13a)

Use the substitution $u = \cos^2 x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4 + \cos^2 x}\,dx$

$$\int_0^{\frac{\pi}{4}} \frac{\sin 2x}{4 + \cos^2 x}dx = -\int_1^{\frac{1}{2}} \frac{du}{4 + u}$$

$$= \Big[\ln(4 + u)\Big]_{\frac{1}{2}}^{1}$$

$$= \ln 5 - \ln\frac{9}{2}$$

$$= \ln\frac{10}{9}$$

$u = \cos^2 x$

$du = -2\cos x \sin x\, dx$

$du = -\sin 2x\, dx$

when $x = 0$, $u = 1$

$x = \frac{\pi}{4}$, $u = \frac{1}{2}$

(v) 2017 Extension 1 HSC Question 11e)

Evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}dx$ using the substitution $x = u^2 - 1$

$$\int_0^3 \frac{x}{\sqrt{x+1}}dx = \int_1^2 \frac{u^2-1}{u} \times 2udu$$

$x = u^2 - 1 \Rightarrow u = \sqrt{x+1}$

$dx = 2udu$

$$= 2\int_1^2 (u^2-1)du$$

when $x = 0$, $u = 1$

$x = 3$, $u = 2$

$$= 2\left[\frac{u^3}{3} - u\right]_1^2$$

$$= 2\left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$$= \frac{8}{3}$$

when substituting
$x = f(u)$
care needs to be taken to ensure the substitution is consistent with the domain of the original function

Exercise 12D; 1, 2ace, 3, 4ace, 5ace etc, 6, 7 & 8ac, 11, 12, 13

Exercise 12E; 1, 2a, 3, 4b, 5ac, 6ace etc, 7ab(i,ii) 8ab (i,iii,vi), 9ab (ii,iv,vi), 10, 12ab, 13