Integrating Quadratic Denominators

$$(1)\int \frac{dx}{a^2 - x^2} = \int \frac{dx}{(a+x)(a-x)} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

$$(2)\int \frac{dx}{x^2 - a^2} = \int \frac{dx}{(x+a)(x-a)} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

(1) and (2) done via partial fractions

$$(3)\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$(4)\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$$

$$(5)\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$$

$$(6)\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$$

e.g. $(i)\int \frac{5dx}{x^2+4x+9}$

$$= \int \frac{5dx}{(x+2)^2+5}$$

$$= 5 \times \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$$= \sqrt{5} \tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right) + c$$

$(ii)\int \frac{(3x+2)dx}{\sqrt{x^2-4x+1}}$

$$= \frac{3}{2}\int \frac{2x-4}{\sqrt{x^2-4x+1}}dx + \int \frac{8dx}{\sqrt{x^2-4x+1}}$$

$$u = x^2-4x+1$$

$$du = (2x-4)dx$$

$$= \frac{3}{2}\int \frac{du}{\sqrt{u}} + \int \frac{8dx}{\sqrt{(x-2)^2-3}}$$

$$= 3\sqrt{u} + 8\log\left|x-2+\sqrt{x^2-4x+1}\right| + c$$

$$= 3\sqrt{x^2-4x+1} + 8\log\left|x-2+\sqrt{x^2-4x+1}\right| + c$$

$$(iii)\int\sqrt{\frac{x+3}{2-x}}dx \quad = \int\sqrt{\frac{x+3}{2-x}}\cdot\sqrt{\frac{x+3}{x+3}}dx$$

$$u = 6-x-x^2$$

$$du = (-2x-1)dx$$

$$= \int\frac{x+3}{\sqrt{6-x-x^2}}dx$$

$$= -\frac{1}{2}\int\frac{-2x-1}{\sqrt{6-x-x^2}}dx + \frac{1}{2}\int\frac{5dx}{\sqrt{6-x-x^2}}$$

$$= -\frac{1}{2}\int\frac{du}{\sqrt{u}} + \frac{5}{2}\int\frac{dx}{\sqrt{\frac{25}{4}-\left(x+\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{2}\times 2\sqrt{u} + \frac{5}{2}\sin^{-1}\left[\frac{2\left(x+\frac{1}{2}\right)}{5}\right] + c$$

$$= -\sqrt{6-x-x^2} + \frac{5}{2}\sin^{-1}\left[\frac{2x+1}{5}\right] + c$$

The "missing" standard integrals

integrand finishes with x

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + c$$

primitive finishes with $a - x$

integrand begins with x

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right] + c$$

primitive begins with $x - a$

Exercise 4E; 2bcef, 3bdf, 4bdef, 5ace, 6ab, 7ac, 8, 9