Inverse Functions

The inverse function of $y = f(x)$ is symbolised $y = f^{-1}(x)$

The composite function of a function and its inverse function is always x i.e. $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

e.g.

$$f(x) = \frac{x-2}{x+2}$$

$$y = \frac{x-2}{x+2} \Rightarrow x = \frac{y-2}{y+2}$$

$$(y+2)x = y-2$$

$$xy + 2x = y - 2$$

$$(x-1)y = -2x-2$$

$$y = \frac{2x+2}{1-x}$$

$$f^{-1}\left(f(x)\right) = \frac{2\left(\frac{x-2}{x+2}\right)+2}{1-\left(\frac{x-2}{x+2}\right)}$$

$$= \frac{2x-4+2x+4}{x+2-x+2}$$

$$= \frac{4x}{4}$$

$$= x$$

$$f\left(f^{-1}(x)\right) = \frac{\left(\frac{2x+2}{1-x}\right)-2}{\left(\frac{2x+2}{1-x}\right)+2}$$

$$= \frac{2x+2-2+2x}{2x+2+2-2x}$$

$$= \frac{4x}{4}$$

$$= x$$

Domain and Range

If (a,b) is a point on $y = f(x)$, then (b,a) is a point on $y = f^{-1}(x)$

thus

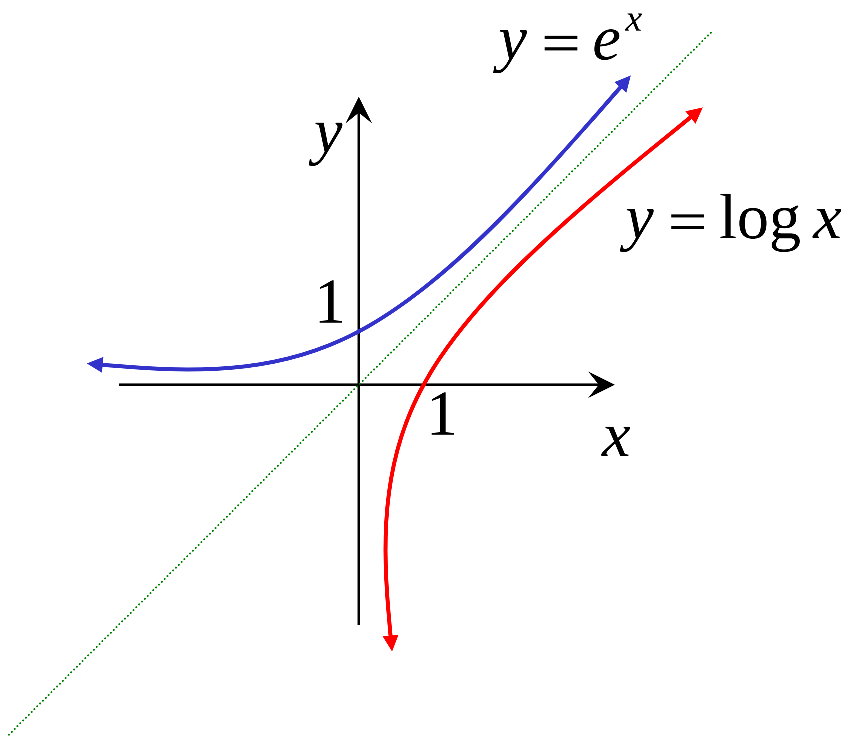
The domain of $y = f(x)$ is the range of $y = f^{-1}(x)$

The range of $y = f(x)$ is the domain of $y = f^{-1}(x)$

e.g. $y = e^x$

Domain: all real x

Range: $y > 0$



$f^{-1}: x = e^y$

$\therefore y = \log x$

Domain: $x > 0$

Range: all real y

Restricting the domain so that the function is one-to-one

A function can be made invertible by restricting the domain so that the function becomes one-to-one.

The domain should be chosen so that the piece of the graph is continually increasing or decreasing.

e.g. $y = x^2$

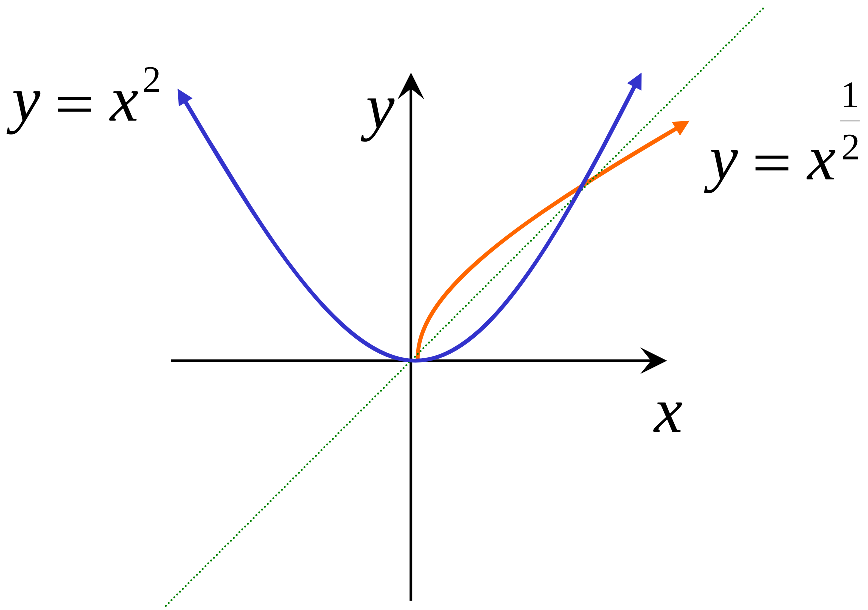
Domain: all real x

Range: $y \geq 0$

NO INVERSE

Restricted Domain: $x \geq 0$

Range: $y \geq 0$



$f^{-1} : x = y^2$

$\therefore y = x^{\frac{1}{2}}$

Domain: $x \geq 0$

Range: $y \geq 0$

Another Way of Finding Range

The domain of an **inverse function** is the range of the original function

e.g. $(i)\ y = x^2$

$$X = Y^2$$

$$Y = \sqrt{X}$$

domain of new function: $X \geq 0$

range of original function: $y \geq 0$

$(ii)\ y = x^2 + 3$

$$X = Y^2 + 3$$

$$Y = \sqrt{X - 3}$$

domain of new function: $X \geq 3$

range of original function: $y \geq 3$

$(iii)\ y = |x + 2| - 5$

$$X = |Y + 2| - 5$$

$$X + 5 = |Y + 2|$$

$$X + 5 \geq 0$$

domain of new function: $X \geq -5$

range of original function: $y \geq -5$

$(iv)\quad y=\sqrt{4-x^2}$

$$X=\sqrt{4-Y^2}$$

$$X^2=4-Y^2$$

$$Y^2=4-X^2$$

$$Y=\sqrt{4-X^2}$$

domain of new function: $-2\leq X\leq 2$

but $y\geq 0$

range of original function: $0\leq y\leq 2$

$(v)\quad y=\dfrac{x+7}{x+4}$

$$X=\frac{Y+7}{Y+4}$$

$$X(Y+4)=Y+7$$

$$XY-Y=7-4X$$

$$Y(X-1)=7-4X$$

$$Y=\frac{7-4X}{X-1}$$

domain of new function: all real X except $X=1$

range of original function: all real y except $y=1$

Exercise 5G: 2ad, 3abc (iii), 4bdefhk, 9, 10a, 11