

Reduction Formula

Reduction (or recurrence) formulae can be used when the integrand is raised to a power.

Integration by parts often used to find the formula.

e.g. (i) Find $\int_1^e x(\log x)^3 dx$

$$I_n = \int_1^e x(\log x)^n dx$$
$$= \frac{1}{2} \left[x^2 (\log x)^n \right]_1^e - \frac{n}{2} \int_1^e x(\log x)^{n-1} dx$$
$$= \frac{e^2}{2} - \frac{n}{2} \int_1^e x(\log x)^{n-1} dx$$
$$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$
$$u = (\log x)^n$$
$$du = \frac{n(\log x)^{n-1}}{x} dx$$
$$v = \frac{1}{2} x^2$$
$$dv = x dx$$

$$\begin{aligned}
\therefore \int_1^e x(\log x)^3 dx &= I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 & \text{OR} & \quad I_0 = \int_1^e x dx = \frac{1}{2} [x^2]_1^e \\
&= \frac{e^2}{2} - \frac{3}{2} \left(\frac{e^2}{2} - I_1 \right) & & \quad = \frac{1}{2} (e^2 - 1) \\
&= -\frac{e^2}{4} + \frac{3}{2}I_1 & & \quad I_1 = \frac{e^2}{2} - \frac{1}{2}I_0 = \frac{e^2}{2} - \frac{1}{2} \times \frac{1}{2} (e^2 - 1) \\
&= -\frac{e^2}{4} + \frac{3}{2} \left(\frac{e^2}{2} - \frac{1}{2}I_0 \right) & & \quad = \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{e^2}{2} - \frac{3}{4}I_0 & & \quad I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{2} - \frac{1}{4} (e^2 + 1) \\
&= \frac{e^2}{2} - \frac{3}{4} \int_1^e x dx & & \quad = \frac{e^2}{4} - \frac{1}{4} \\
&= \frac{e^2}{2} - \frac{3}{8} [x^2]_1^e & & \quad I_3 = \frac{e^2}{2} - \frac{3}{2}I_2 = \frac{e^2}{2} - \frac{3}{2} \times \frac{1}{4} (e^2 - 1) \\
&= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8} = \underline{\underline{\frac{e^2}{8} + \frac{3}{8}}} & & \quad = \underline{\underline{\frac{e^2}{8} + \frac{3}{8}}}
\end{aligned}$$

Integration by parts is the commonest way of getting reduction formulae, but it is not the only method.

Some just involve the use of a **trig identity**.

(ii) Given that $I_n = \int \cot^n x dx$, find I_6

$$\begin{aligned} I_n &= \int \cot^n x dx \\ &= \int \cot^{n-2} x \cot^2 x dx \\ &= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) dx \\ &= \int \cot^{n-2} x \operatorname{cosec}^2 x dx - \int \cot^{n-2} x dx \\ &= -\int u^{n-2} du - I_{n-2} \\ &= -\frac{1}{n-1} u^{n-1} - I_{n-2} \\ &= -\frac{1}{n-1} \cot^{n-1} x - I_{n-2} \end{aligned}$$

$$\int \cot^n x dx$$

Uses the same technique, for all powers.

So just use that technique

$$u = \cot x$$

$$du = -\operatorname{cosec}^2 x dx$$

OR

Reduction formulae involving trig often reduces by 2. If you do not have to use parts you could try $I_n \pm I_{n+2}$

$$\begin{aligned}I_n + I_{n+2} &= \int \cot^n x dx + \int \cot^{n+2} x dx \\&= \int (\cot^n x + \cot^{n+2} x) dx \\&= \int \left\{ \cot^n x (1 + \cot^2 x) \right\} dx \\&= \int \cot^n x \operatorname{cosec}^2 x dx \\&= -\frac{\cot^{n+1} x}{n+1}\end{aligned}$$

$$I_{n+2} = -\frac{\cot^{n+1} x}{n+1} - I_n$$

$$I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\begin{aligned}\int \cot^6 x dx &= I_6 \\ &= -\frac{1}{5} \cot^5 x - I_4 \\ &= -\frac{1}{5} \cot^5 x + \frac{1}{3} \cot^3 x + I_2 \\ &= -\frac{1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - I_0 \\ &= -\frac{1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - \int dx \\ &= -\frac{1}{5} \cot^5 x + \frac{1}{3} \cot^3 x - \cot x - x + c\end{aligned}$$

Sometimes integration by parts is not enough by itself.

Some involve the use of **parts** along with **algebraic manipulation** or use of a **trig identity**.

(iii) Given that $I_n = \int (1+x^2)^n dx$, find a reduction formula

connecting I_n with I_{n-1}

$$\begin{aligned} I_n &= \int (1+x^2)^n dx & u &= (1+x^2)^n & v &= x \\ &= x(1+x^2)^n - 2n \int x^2 (1+x^2)^{n-1} dx & du &= 2nx(1+x^2)^{n-1} dx & dv &= dx \\ &= x(1+x^2)^n - 2n \int (1+x^2-1)(1+x^2)^{n-1} dx \\ &= x(1+x^2)^n - 2n \int (1+x^2)(1+x^2)^{n-1} dx + 2n \int (1+x^2)^{n-1} dx \\ &= x(1+x^2)^n - 2n \int (1+x^2)^n dx + 2n \int (1+x^2)^{n-1} dx \\ &= x(1+x^2)^n - 2nI_n + 2nI_{n-1} \\ (2n+1)I_n &= x(1+x^2)^n + 2nI_{n-1} & \therefore I_n &= \frac{x(1+x^2)^n}{(2n+1)} + \frac{2n}{(2n+1)} I_{n-1} \end{aligned}$$

e.g. (iv) (1987)

Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, prove that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$

$$\begin{aligned}
 I_n &= \int_0^{\frac{\pi}{2}} \cos^n x dx && \text{where } n \text{ is an integer and } n \geq 2, \text{ hence evaluate } \int_0^{\frac{\pi}{2}} \cos^5 x dx \\
 &= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x dx && \begin{aligned} u &= \cos^{n-1} x && v = \sin x \\ du &= -(n-1)\cos^{n-2} x \sin x dx && dv = \cos x dx \end{aligned} \\
 &= \left[\cos^{n-1} x \sin x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx \\
 &= \left\{ \cos^{n-1} \frac{\pi}{2} \sin \frac{\pi}{2} - \cos^{n-1} 0 \sin 0 \right\} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x (1 - \cos^2 x) dx \\
 &= (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2} x dx - (n-1) \int_0^{\frac{\pi}{2}} \cos^n x dx && \therefore nI_n = (n-1)I_{n-2} \\
 &= (n-1)I_{n-2} - (n-1)I_n && \underline{I_n = \left(\frac{n-1}{n}\right)I_{n-2}}
 \end{aligned}$$

$$I_{n-2} - I_n$$

$$= \int_0^{\frac{\pi}{2}} (\cos^{n-2} x - \cos^n x) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x (1 - \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x dx$$

$$= \frac{-1}{n-1} \left[\sin x \cos^{n-1} x \right]_0^{\frac{\pi}{2}} + \frac{1}{n-1} \int_0^{\frac{\pi}{2}} \cos^n x dx$$

$$= \frac{1}{n-1} I_n$$

OR

$$u = \sin x$$

$$du = \cos x dx$$

$$v = \frac{-\cos^{n-1} x}{n-1}$$

$$dv = \cos^{n-2} x \sin x dx$$

$$\therefore (n-1)I_{n-2} - (n-1)I_n = I_n$$

$$(n-1)I_{n-2} = nI_n$$

$$\underline{I_n = \left(\frac{n-1}{n} \right) I_{n-2}}$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^5 x dx &= I_5 \\ &= \frac{4}{5} I_3 \\ &= \frac{4}{5} \times \frac{2}{3} I_1 \\ &= \frac{8}{15} \int_0^{\frac{\pi}{2}} \cos x dx \\ &= \frac{8}{15} [\sin x]_0^{\frac{\pi}{2}} \\ &= \frac{8}{15} \left(\sin \frac{\pi}{2} - \sin 0 \right) \\ &= \frac{8}{15}\end{aligned}$$

(v) (2004 Question 8b)

Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2, \dots$

a) Show that $I_n + I_{n+2} = \frac{1}{n+1}$

$$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n dx + \int_0^{\frac{\pi}{4}} \tan^{n+2} dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x (1 + \tan^2 x) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^n x \sec^2 x dx$$

$$= \int_0^1 u^n du$$

$$= \left[\frac{u^{n+1}}{n+1} \right]_0^1$$

$$= \frac{1}{n+1} - 0 = \frac{1}{n+1}$$

$$u = \tan x$$

$$du = \sec^2 x dx$$

when $x = 0, u = 0$

$$x = \frac{\pi}{4}, u = 1$$

b) Deduce that $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ for $n \geq 1$

$$J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$$

$$= (-1)^n I_{2n} + (-1)^n I_{2n-2}$$

$$= (-1)^n (I_{2n} + I_{2n-2})$$

$$= \frac{(-1)^n}{2n-2+1}$$

(using part a) let $n = 2n - 2$)

$$= \frac{(-1)^n}{\underline{2n-1}}$$

c) Show that $J_m = \frac{\pi}{4} + \sum_{n=1}^m \frac{(-1)^n}{2n-1}$

$$J_m - J_{m-1} = \frac{(-1)^m}{2m-1}$$

$$J_m = \frac{(-1)^m}{2m-1} + J_{m-1}$$

$$= \frac{(-1)^m}{2m-1} + \frac{(-1)^{m-1}}{2m-3} + J_{m-2}$$

$$= \frac{(-1)^m}{2m-1} + \frac{(-1)^{m-1}}{2m-3} + \dots + \frac{(-1)^1}{1} + J_0$$

$$= \sum_{n=1}^m \frac{(-1)^n}{2n-1} + \int_0^{\frac{\pi}{4}} dx$$

$$= \sum_{n=1}^m \frac{(-1)^n}{2n-1} + [x]_0^{\frac{\pi}{4}}$$

$$= \sum_{n=1}^m \frac{(-1)^n}{2n-1} + \frac{\pi}{4}$$

d) Use the substitution $u = \tan x$ to show that $I_n = \int_0^1 \frac{u^n}{1+u^2} du$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$$

$$u = \tan x \Rightarrow x = \tan^{-1} u$$

$$I_n = \int_0^1 u^n \times \frac{du}{1+u^2}$$

$$dx = \frac{du}{1+u^2}$$

$$\underline{I_n = \int_0^1 \frac{u^n du}{1+u^2}}$$

$$\text{when } x = 0, u = 0$$

$$x = \frac{\pi}{4}, u = 1$$

e) Deduce that $0 \leq I_n \leq \frac{1}{n+1}$ and conclude that $J_n \rightarrow 0$ as $n \rightarrow \infty$

$$\frac{u^n}{1+u^2} \geq 0, \text{ for all } u \geq 0$$

$$\therefore I_n = \int_0^1 \frac{u^n}{1+u^2} du \geq 0, \text{ for all } u \geq 0$$

$$I_n + I_{n+2} = \frac{1}{n+1}$$

$$I_n = \frac{1}{n+1} - I_{n+2}$$

$$\therefore I_n \leq \frac{1}{n+1}, \text{ as } I_{n+2} \geq 0$$

$$\therefore 0 \leq I_n \leq \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$$

$$\therefore \lim_{n \rightarrow \infty} I_n = 0$$

$$\begin{aligned} \lim_{n \rightarrow \infty} J_n &= \lim_{n \rightarrow \infty} (-1)^n I_{2n} \\ &= \underline{0} \end{aligned}$$

**Exercise 4H; 2, 5, 6,
7, 8, 9, 10, 12, 14, 17**