Dot Product

A vector is a line segment in both 2D & 3D, the rules and properties remain the same in both dimensions.

$$\text{If } \underset{\sim}{u} \neq 0 \wedge \underset{\sim}{v} \neq 0$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

NOTE: θ is acute or obtuse

$$\text{If } \underset{\sim}{u} = 0 \vee \underset{\sim}{v} = 0$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$$

$$\text{If } \underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \text{ and } \underset{\sim}{v} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Dot Product Properties

(1) $-\,|\underset{\sim}{u}||\underset{\sim}{v}| \leq \underset{\sim}{u} \cdot \underset{\sim}{v} \leq |\underset{\sim}{u}||\underset{\sim}{v}|$

(2) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$

(3) $$\begin{aligned} \underset{\sim}{u} \cdot \underset{\sim}{u} &= x_1{}^2 + y_1{}^2 \\ &= |\underset{\sim}{u}|^2 \end{aligned}$$

(4) $\underset{\sim}{a} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{a} \cdot \underset{\sim}{u} + \underset{\sim}{a} \cdot \underset{\sim}{v}$

(5) $$\begin{aligned} (\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) &= \underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{v} \\ &= |\underset{\sim}{u}|^2 - |\underset{\sim}{v}|^2 \end{aligned}$$

(6) $\lambda \underset{\sim}{u} \cdot \underset{\sim}{v} = \lambda(\underset{\sim}{u} \cdot \underset{\sim}{v})$

(7) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \iff \underset{\sim}{u} \perp \underset{\sim}{v}$

(8) $\underset{\sim}{u} \cdot \underset{\sim}{v} = \pm|\underset{\sim}{u}||\underset{\sim}{v}| \iff \underset{\sim}{u} \parallel \underset{\sim}{v}$

(9) $|\underset{\sim}{u}||\underset{\sim}{v}| > 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have the same direction

$|\underset{\sim}{u}||\underset{\sim}{v}| < 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have opposite directions eg (i) Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be three 3-dimensional vectors.

Prove that $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c})$$

$$= \left(a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}\right) \cdot \left[(b_1 + c_1)\underset{\sim}{i} + (b_2 + c_2)\underset{\sim}{j} + (b_3 + c_3)\underset{\sim}{k}\right]$$

$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$$

$$= a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$= \underline{\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}}$$

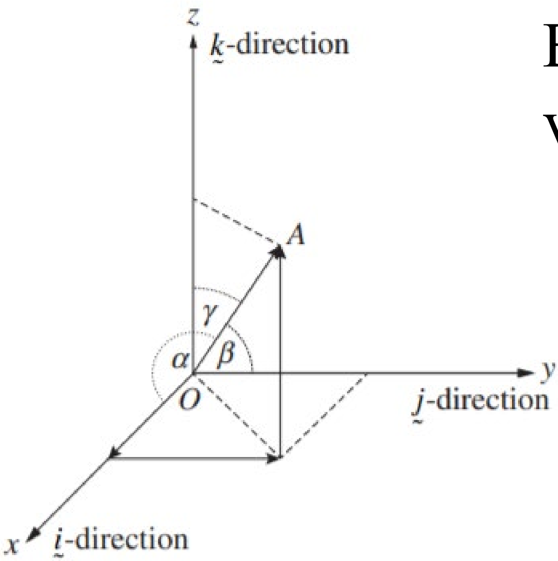
(ii) Prove that the vectors $3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}$ are perpendicular

$$\left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right) \cdot \left(-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}\right) = (3)(-4) + (-2)(-8) + (4)(-1)$$

$$= 0$$

$$\underline{\therefore \left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right) \perp \left(-4\underset{\sim}{i} - 8\underset{\sim}{j} - \underset{\sim}{k}\right)}$$

(iii) The point A has (non-zero) position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and the vector $\overrightarrow{OA}$ makes angles α, β and γ with the x, y and z axes respectively



By taking a dot product with the three unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Exercise 5C;
1a, 4, 6, 9, 11, 14b, 15b, 16a, 18, 20, 21

let $\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$;

$\underset{\sim}{u} \cdot \underset{\sim}{i} = a$

also $\underset{\sim}{u} \cdot \underset{\sim}{i} = |\underset{\sim}{u}||\underset{\sim}{i}|\cos\alpha \ = \sqrt{a^2 + b^2 + c^2}\cos\alpha$

$\therefore \quad a = \sqrt{a^2 + b^2 + c^2}\cos\alpha$

Similarly; $\cos^2\beta = \dfrac{b^2}{a^2 + b^2 + c^2}$

$$\cos^2\alpha = \frac{a^2}{a^2 + b^2 + c^2}$$

$$\cos^2\gamma = \frac{c^2}{a^2 + b^2 + c^2}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a \ + b^2 + c^2}{a^2 + b^2 + c^2}$$

$$=1$$