Applications of the Dot Product

Angle Between Two Vectors

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

e.g. Find the angle between the vectors $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\2\\1\end{pmatrix}$

$$\cos\theta = \frac{-4+4+3}{\sqrt{1^2+2^2+3^2}\sqrt{(-4)^2+2^2+1^2}}$$

$$= \frac{3}{\sqrt{14}\sqrt{21}}$$

$$= \frac{3}{7\sqrt{6}}$$

$\therefore$ the angle between the two vectors is 80° to the nearest degree

Vector Projections

$$\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\underset{\sim}{v}\cdot\underset{\sim}{v}}\right)\underset{\sim}{v}$$

(*ii*) Find the length of the projection of $\underset{\sim}{u} = 4\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$ onto $\underset{\sim}{v} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$$\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{v}|} = \frac{8 - 10 - 3}{\sqrt{2^2 + 2^2 + 1^2}}$$

$$= -\frac{5}{3}$$

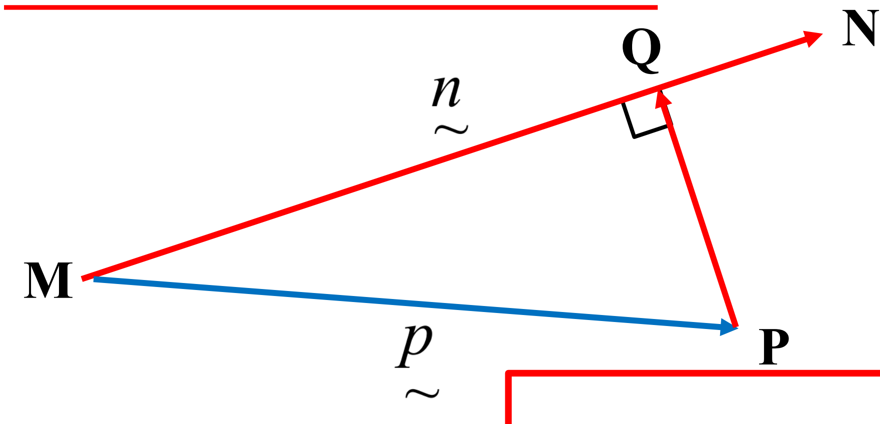
(*iii*) Find the projection of $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$ onto $\underset{\sim}{v} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$$\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \left(\frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\underset{\sim}{v}\cdot\underset{\sim}{v}}\right)\underset{\sim}{v} = \frac{2 + 3 - 8}{1^2 + 1^2 + 2^2}\left(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \frac{3}{4}\left(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}\right)$$

$$= \frac{3}{4}\underset{\sim}{i} + \frac{3}{4}\underset{\sim}{j} + \frac{3}{2}\underset{\sim}{k}$$

Perpendicular Distance



$$\overrightarrow{MQ} = \text{proj}_{\underset{\sim}{n}} \underset{\sim}{p}$$

$$\overrightarrow{MQ} = \overrightarrow{MP} + \overrightarrow{PQ}$$

$$\overrightarrow{PQ} = \overrightarrow{MQ} - \overrightarrow{MP}$$

$$= \text{proj}_{\underset{\sim}{n}} \underset{\sim}{p} - \underset{\sim}{p}$$

$$\text{distance} = \left| \text{proj}_{\underset{\sim}{n}} \underset{\sim}{p} - \underset{\sim}{p} \right|$$

(iv) Find the perpendicular distance from $A(1,0,2)$ to the line joining $B(2,2,1)$ and $C(3,1,1)$

Let $\underset{\sim}{p} = \overrightarrow{BA} = -\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ $\qquad \underset{\sim}{u} = \overrightarrow{BC} = \underset{\sim}{i} - \underset{\sim}{j}$

$$\text{distance} = \left| \text{proj}_{\underset{\sim}{u}} \underset{\sim}{p} - \underset{\sim}{p} \right|$$

$$= \left| \frac{1}{2}\left(\underset{\sim}{i} - \underset{\sim}{j}\right) - \left(-\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right) \right|$$

$$= \left| \frac{3}{2}\underset{\sim}{i} + \frac{3}{2}\underset{\sim}{j} - \underset{\sim}{k} \right| \quad = \frac{\sqrt{22}}{2} \text{ units}$$

(v) Classify the triangle formed by joining the points $A(-1,3,3)$, $B(2,5,4)$ and $C(0,3,2)$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad \left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$\overrightarrow{BC} = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} \qquad \left|\overrightarrow{\mathrm{BC}}\right| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12}$$

$$\overrightarrow{CA} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \qquad \left|\overrightarrow{\mathrm{CA}}\right| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

$$\left|\overrightarrow{\mathrm{CA}}\right|^2 + \left|\overrightarrow{\mathrm{BC}}\right|^2 = \left|\overrightarrow{\mathrm{AB}}\right|^2$$

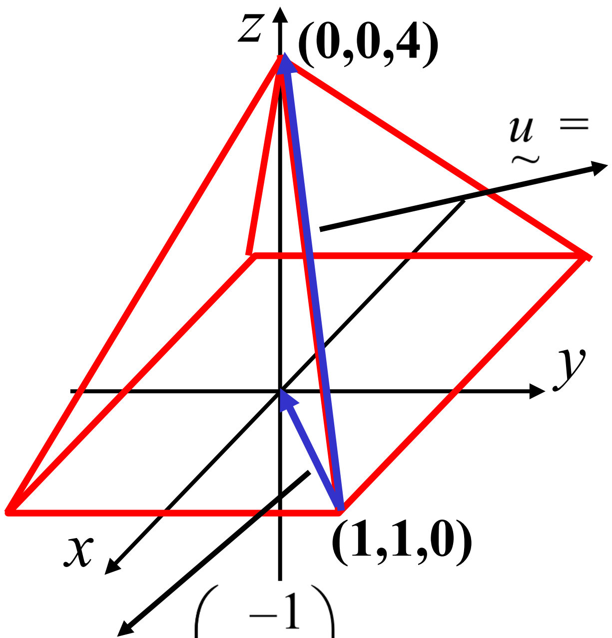
$\therefore$ ΔABC is aright-angled triangle

OR

$$\overrightarrow{\mathrm{BC}} \cdot \overrightarrow{\mathrm{CA}} = \frac{2 + 0 - 2}{\sqrt{12}\,\sqrt{2}} = 0$$

thus $\overrightarrow{\mathrm{BC}} \perp \overrightarrow{\mathrm{CA}}$

(*vi*) *ABCD* is the base of a square pyramid of side 2 units, and *V* is the vertex. The pyramid is symmetrical and has a height of 4 units. Calculate the acute angle between one of the slant edges and the base of the pyramid, to the nearest degree.



$$\underset{\sim}{u} = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} \qquad \underset{\sim}{v} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$\cos\theta = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

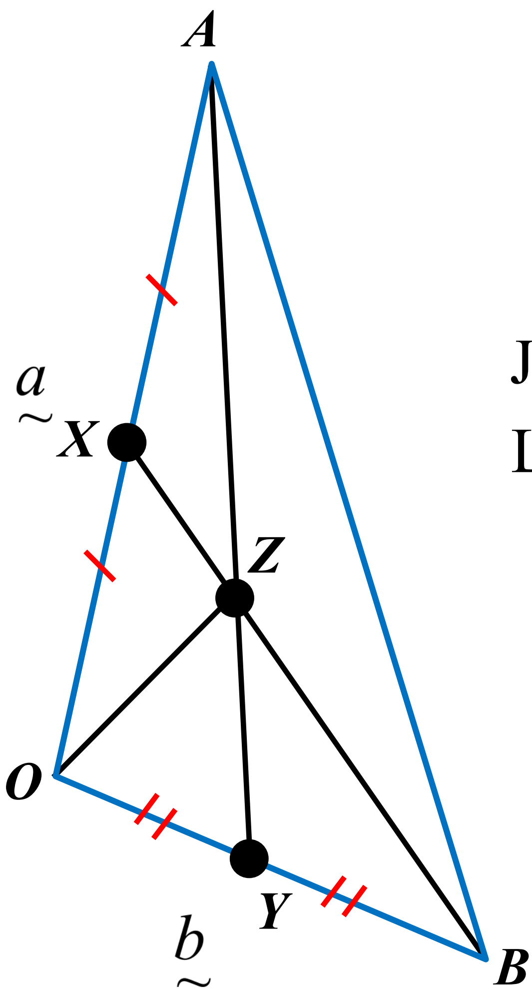
$$\cos\theta = \frac{1 + 1 + 0}{\sqrt{1^2 + 1^2 + 4^2}\sqrt{1^2 + 1^2 + 0^2}}$$

$$= \frac{2}{\sqrt{36}}$$

$$= \frac{1}{3}$$

$\theta = 71°$ (to the nearest degree)

(*vii*) The medians of a triangle meet at a point. Prove that the medians divide each other in the ratio 2:1.



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$

$$\overrightarrow{OX} = \frac{1}{2}\underset{\sim}{a} \quad \overrightarrow{OY} = \frac{1}{2}\underset{\sim}{b}$$

Join AY and BX meeting at Z ,let $\overrightarrow{OZ} = \underset{\sim}{z}$

Let Z divide BX in the ratio m:n

$$\frac{\overrightarrow{AZ}}{\overrightarrow{ZY}} = \frac{m}{n}$$

$$n\overrightarrow{AZ} = m\overrightarrow{ZY}$$

$$n(\underset{\sim}{z} - \underset{\sim}{a}) = m\left(\frac{\underset{\sim}{b}}{2} - \underset{\sim}{z}\right)$$

$$(m + n)\underset{\sim}{z} = n\underset{\sim}{a} + \frac{m\underset{\sim}{b}}{2}$$

Similarly Z divides AY in the ratio $m:n$

$$\text{i.e } (m+n)\underset{\sim}{z} = n\underset{\sim}{b} + \frac{m\underset{\sim}{a}}{2}$$

$$\therefore\ n\underset{\sim}{a} + \frac{m\underset{\sim}{b}}{2} = n\underset{\sim}{b} + \frac{m\underset{\sim}{a}}{2}$$

$$\frac{m}{2}(\underset{\sim}{b} - \underset{\sim}{a}) = n(\underset{\sim}{b} - \underset{\sim}{a})$$

$$\frac{m}{2} = n$$

$$\frac{m}{n} = \frac{2}{1}$$

Thus the medians divide each other in the ratio 2:1

Exercise 5D; 1, 3, 4b, 6a, 7b, 10, 11, 12, 13a, 14, 16, 18, 20, 21

Exercise 5E; 1, 3, 5, 8, 9, 11