Inverse Functions

If there exists a one-to-one relationship between the two sets, then both the relation and the inverse relation are functions.
In this situation the inverse relation is called the **inverse function.**

A function is either one-to-one or many-to-one.

If the function is one-to-one, then the composite function of a function and its inverse is always x

i.e. $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

Testing For Inverse Functions

(1) Use a horizontal line test

OR

(2) When $x = f(y)$ is rewritten as $y = g(x)$, $y = g(x)$ is unique.

The inverse function of an odd function will always be odd
Even functions do not have inverse functions

Domain and Range

If (a, b) is a point on $y = f(x)$, then (b, a) is a point on $y = f^{-1}(x)$

thus

The domain of $y = f(x)$ is the range of $y = f^{-1}(x)$

The range of $y = f(x)$ is the domain of $y = f^{-1}(x)$

e.g. $y = e^x$

Domain: all real x

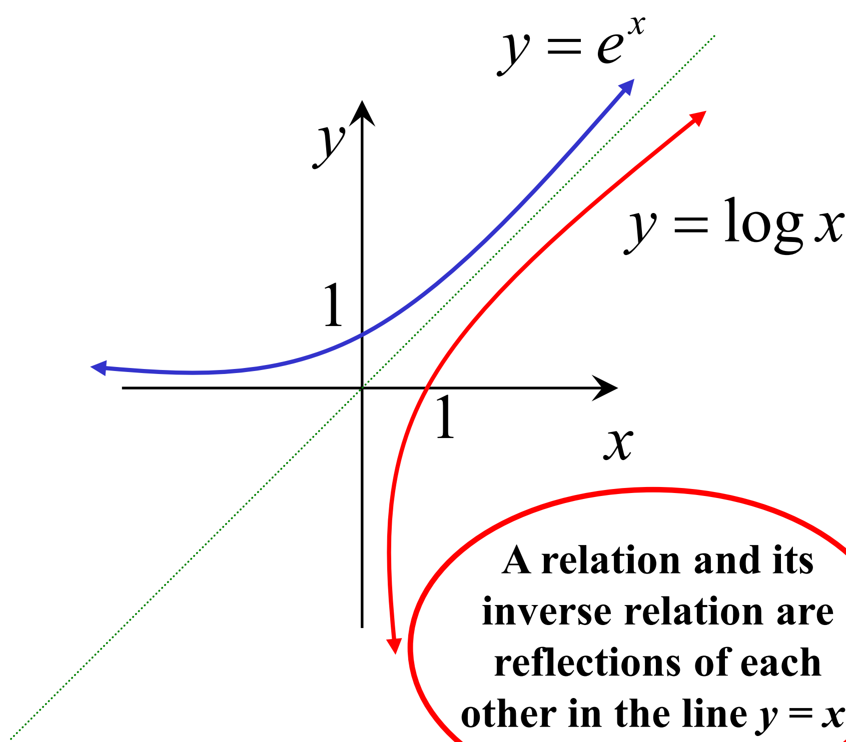
Range: $y > 0$

$f^{-1} : x = e^y$

$\therefore y = \log x$

Domain: $x > 0$

Range: all real y



A relation and its inverse relation are reflections of each other in the line $y = x$.

Restricting the domain so that the function is one-to-one

A function can be made invertible by restricting the domain so that the function becomes one-to-one.

The domain should be chosen so that the piece of the graph is continually increasing or decreasing.

e.g. $y = x^2$

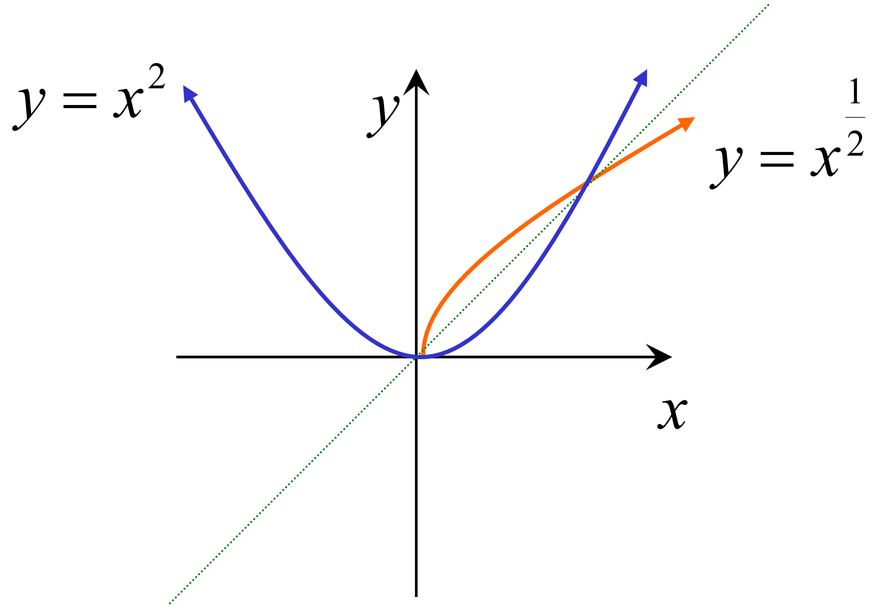
Domain: all real x

Range: $y \geq 0$

NO INVERSE

Restricted Domain: $x \geq 0$

Range: $y \geq 0$



$f^{-1} : x = y^2$

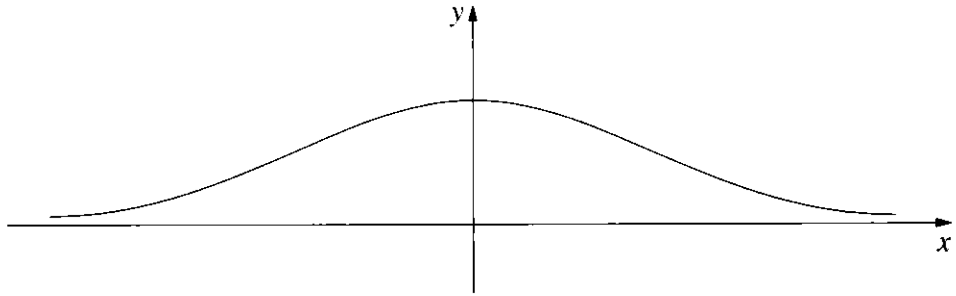
$\therefore y = x^{\frac{1}{2}}$

Domain: $x \geq 0$

Range: $y \geq 0$

(iv) **2010 HSC Question 3b)**

Let $f(x) = e^{-x^2}$. The diagram shows the graph of $y = f(x)$



a) The graph has two points of inflection. Find the x coordinates of these points.

Possible inflection points occur when $f''(x) = 0$

$$f(x) = e^{-x^2}$$

$$f'(x) = -2xe^{-x^2}$$

$$f''(x) = (-2x)\left(-2xe^{-x^2}\right) + \left(e^{-x^2}\right)(-2)$$

$$= \left(4x^2 - 2\right)e^{-x^2}$$

$$f''(x) = 0$$

$$e^{-x^2} = 0$$

$$4x^2 - 2 = 0$$

$$x = \pm\frac{1}{\sqrt{2}}$$

b) Explain why the domain of $f(x)$ must be restricted if $f(x)$ is to have an inverse function.

In order to have an inverse function there must exist a **one-to-one** relationship between the domain and range. i.e. every value in the domain gives a unique value in the range.

c) Find a formula for $f^{-1}(x)$ if the domain of $f(x)$ is restricted to $x \geq 0$

$$x = e^{-y^2}$$

$$-y^2 = \ln x$$

$$y^2 = -\ln x$$

$$= \ln\left(\frac{1}{x}\right)$$

$$y = \pm\sqrt{\ln\left(\frac{1}{x}\right)}$$

However the domain of the original function becomes the range of the inverse function.

i.e. $y \geq 0$

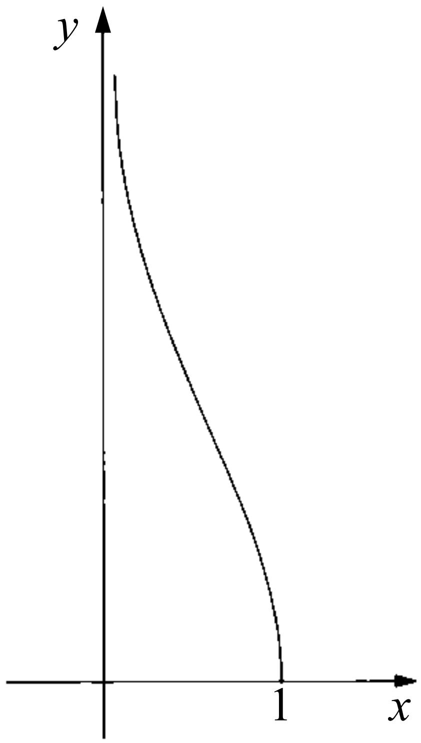
$$y = \sqrt{\ln\left(\frac{1}{x}\right)}$$

d) State the domain of $f^{-1}(x)$

The range of the original function becomes the domain of the inverse function.

$\therefore 0 < x \leq 1$

e) Sketch the curve $y = f^{-1}(x)$



Exercise 17A; 1, 3c, 4, 5bc, 6, 8, 9, 10, 11, 13, 15, 16, 18, 20