Complex Numbers & Trig Identities

(i) Express $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$$
$$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$$

By equating real and imaginary parts

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

(ii) Express $\sin 3\theta, \cos 3\theta$ and $\tan 3\theta$ in terms of $\sin\theta, \cos\theta$ and $\tan\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta$$

By equating real and imaginary parts

$$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$$
$$= 3\sin\theta\left(1-\sin^2\theta\right) - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$$
$$= \cos^3\theta - 3\left(1-\cos^2\theta\right)\cos\theta$$
$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$$
$$= \frac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$$
$$= \frac{\dfrac{3\sin\theta\cos^2\theta - \sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta - 3\sin^2\theta\cos\theta}{\cos^3\theta}}$$
$$= \frac{\dfrac{3\sin\theta}{\cos\theta} - \dfrac{\sin^3\theta}{\cos^3\theta}}{\dfrac{\cos^3\theta}{\cos^3\theta} - \dfrac{3\sin^2\theta}{\cos^2\theta}} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(iii) Show that $x=\cos\frac{\pi}{9}$ is a solution to $8x^3-6x-1=0$

$$8x^3-6x-1=0$$

$$2(4x^3-3x)-1=0$$

let $x=\cos\theta$

$$2(4\cos^3\theta-3\cos\theta)-1=0$$

$$2\cos 3\theta-1=0$$

$$\cos 3\theta=\frac{1}{2}$$

$$3\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3}$$

$$\theta=\frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9}$$

We require three consecutive **unique** solutions

$$x=\cos\frac{\pi}{9},\ \cos\frac{5\pi}{9},\ \cos\frac{7\pi}{9}$$

$\therefore x=\cos\frac{\pi}{9}$ is a solution

(iii) Evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{5\pi}{9}$

$$x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9} \text{ are the roots of } 8x^3 - 6x - 1 = 0$$

$$\alpha\beta\gamma = \frac{1}{8}$$ product of roots

$$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$$

but $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

$$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{5\pi}{9} = -\frac{1}{8}$$

(iv) If $z=\cos\theta+i\sin\theta$, find $z^n+\dfrac{1}{z^n}$ and $z^n-\dfrac{1}{z^n}$

$$z^n=\cos n\theta+i\sin n\theta$$

$$\frac{1}{z^n}=z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$$

$$=\cos n\theta-i\sin n\theta \qquad \left\{\begin{array}{l}\cos \text{ is even function} \Rightarrow \cos(-x)=\cos x\\ \sin \text{ is odd function} \Rightarrow \sin(-x)=-\sin x\end{array}\right\}$$

$$z^n+\frac{1}{z^n}=2\cos n\theta \qquad\qquad z^n-\frac{1}{z^n}=2i\sin n\theta$$

(v) Express $\cos^3\theta$ in terms of $\cos n\theta$

$$(2\cos\theta)^3=\left(z+\frac{1}{z}\right)^3$$

$$8\cos^3\theta=z^3+3z+\frac{3}{z}+\frac{1}{z^3}$$

$$=\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)$$

$$8\cos^3\theta=2\cos3\theta+6\cos\theta$$

$$\therefore \cos^3\theta=\frac{1}{4}\cos3\theta+\frac{3}{4}\cos\theta$$

Exercise 3B; 2 to 4, 6 to 8, 11 to 14