Geometric Proofs

Things to keep in mind when using vectors in a geometric proof

 * choose a vertex or other key point to represent the origin

 * sum (or difference) of vectors creates a triangle

 * parallel vectors are multiples of each other

 * the angle between vectors can be found using the dot product

 * perpendicular vectors have a dot product equal to zero

 * vectors can be written as their position vector using *"head minus tail"*

e.g. Prove that the diagonals of a rhombus are perpendicular

$OABC$ is a rhombus

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$

Diagonals are OB and AC

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

$\overrightarrow{OB} \cdot \overrightarrow{AC} = (\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a})$

$= \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2$
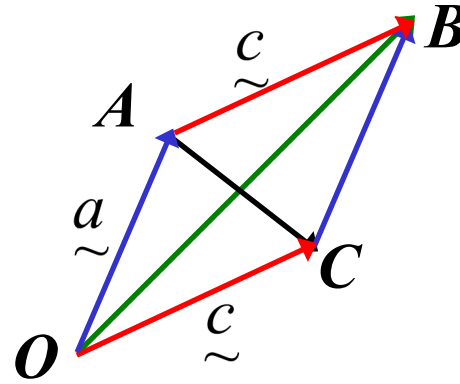
however $|\underset{\sim}{a}| = |\underset{\sim}{c}|$ (sides in a rhombus are =)

so $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

$\therefore OB \perp AC$

(*ii*) Prove Pythagoras' Theorem

OAB is a right angled triangle

Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$OA^2 + OB^2$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$
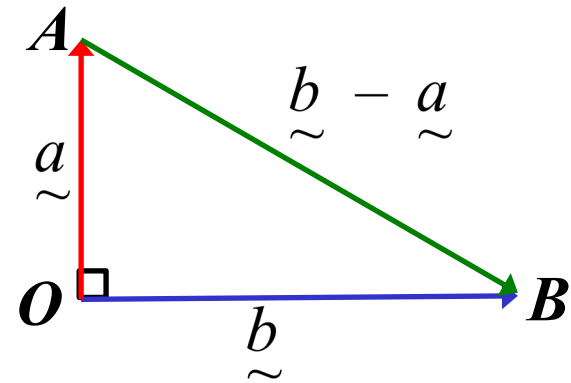
$AB^2 = |\underset{\sim}{b} - \underset{\sim}{a}|^2$

$= (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{b} \cdot \underset{\sim}{b} - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

however $OA \perp OB$

so $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$AB^2 = \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= |\underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2$

$\therefore AB^2 = OA^2 + OB^2$

(*iii*) *ABCD* is a quadrilateral, *P, Q, R* and *S* are midpoints of the lines *AC, BD, AD* and *BC* respectively.

What type of quadrilateral is *PRQS*?

treat *A* as the origin

Let $\overrightarrow{AB} = \underset{\sim}{u}$, $\overrightarrow{AC} = \underset{\sim}{v}$, $\overrightarrow{AD} = \underset{\sim}{w}$

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\underset{\sim}{v}$

$\overrightarrow{BQ} = \frac{1}{2}\overrightarrow{BD} = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{u})$

$\overrightarrow{AR} = \frac{1}{2}\overrightarrow{AD} = \frac{1}{2}\underset{\sim}{w}$

$\overrightarrow{BS} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$

$\overrightarrow{PR} = \overrightarrow{PA} + \overrightarrow{AR}$
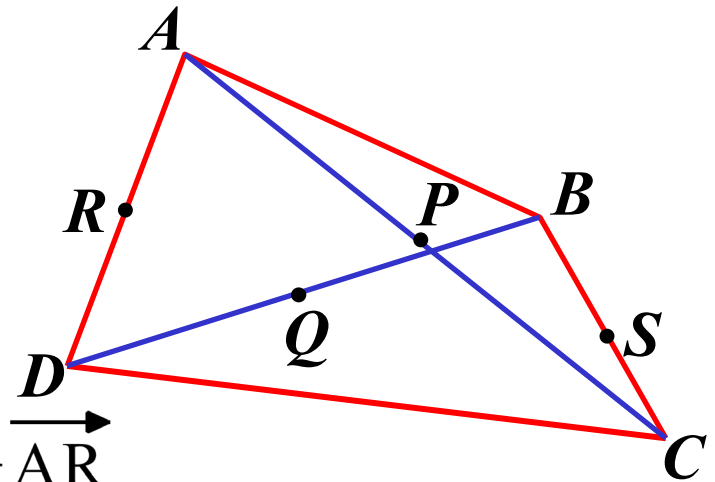
$= -\frac{1}{2}\underset{\sim}{v} + \frac{1}{2}\underset{\sim}{w} = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{v})$

$\overrightarrow{SQ} = \overrightarrow{SB} + \overrightarrow{BQ}$

$= \frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v}) + \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{u}) = \frac{1}{2}(\underset{\sim}{w} - \underset{\sim}{v})$
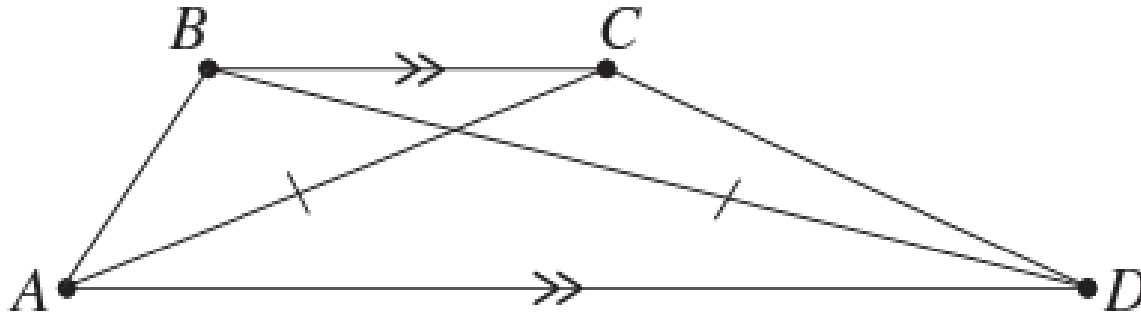
$\therefore \overrightarrow{PR} = \overrightarrow{SQ}$

Thus *PRQS* is a parallelogram as a pair of opposite sides are both equal and parallel

(iv) 2021 Extension 1 HSC Q14c) (ii)

In the trapezium *ABCD, BC* is parallel to *AD* and $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{BD}\right|$

Let $\underset{\sim}{a} = \overrightarrow{AB}$, $\underset{\sim}{b} = \overrightarrow{BC}$ and $\overrightarrow{AD} = k\overrightarrow{BC}$, where $k > 0$

Show that $2\underset{\sim}{a}\cdot\underset{\sim}{b} + (1-k)|\underset{\sim}{b}|^2 = 0$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \qquad\qquad \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$$

$$= \underset{\sim}{a} + \underset{\sim}{b} \qquad\qquad\qquad = -\underset{\sim}{a} + k\underset{\sim}{b}$$

$$\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{BD}\right|^2$$

$$(\underset{\sim}{a} + \underset{\sim}{b})\cdot(\underset{\sim}{a} + \underset{\sim}{b}) = (k\underset{\sim}{b} - \underset{\sim}{a})\cdot(k\underset{\sim}{b} - \underset{\sim}{a})$$

$$|\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2 = k^2|\underset{\sim}{b}|^2 - 2k\underset{\sim}{b}\cdot\underset{\sim}{a} + |\underset{\sim}{a}|^2$$

$$|\underset{\sim}{a}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2 = k^2|\underset{\sim}{b}|^2 - 2k\underset{\sim}{b} \cdot \underset{\sim}{a} + |\underset{\sim}{a}|^2$$

$$2(k+1)\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k^2)|\underset{\sim}{b}|^2 = 0$$

$$2(k+1)\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)(1+k)|\underset{\sim}{b}|^2 = 0$$

$$\underline{2\underset{\sim}{a} \cdot \underset{\sim}{b} + (1-k)|\underset{\sim}{b}|^2 = 0} \qquad (k > 0 \;\therefore\; 1+k \neq 0)$$

Exercise 8D; 1 to 4, 6 to 8, 10, 11, 12