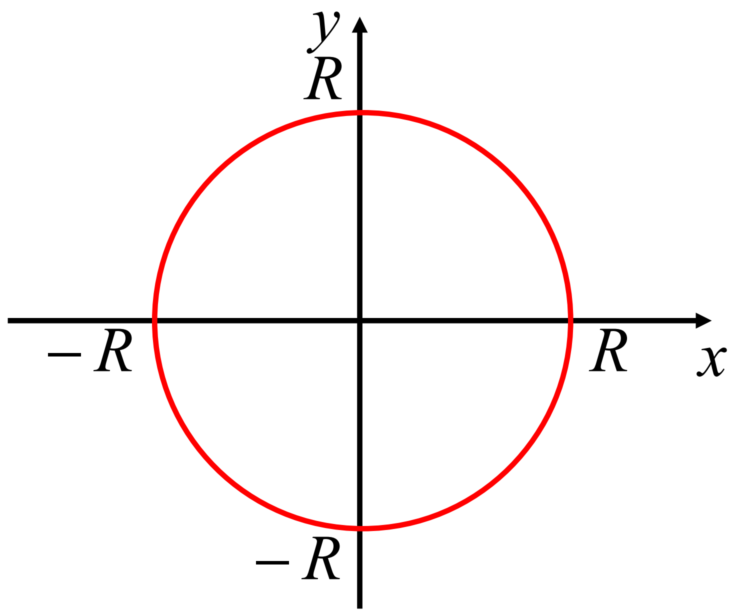
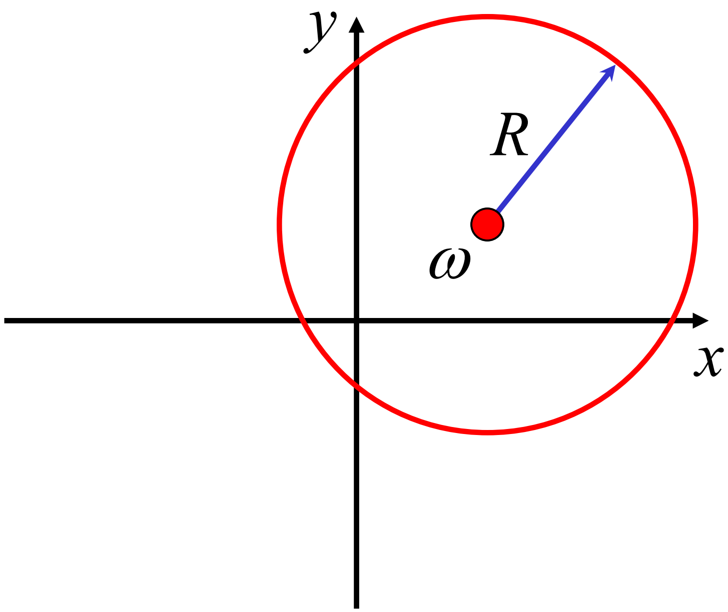
Locus and Complex Numbers

Circles





$$z\bar{z} = R^2$$

or

$$|z| = R$$

$$(z-\omega)(\bar{z}-\bar{\omega}) = R^2$$

or

$$|z-\omega| = R$$

e.g. (i) Express these circles in terms of z

a) $x^2 + y^2 = 16$

$$|z| = 4$$

$$(z\bar{z} = 16)$$

b) $x^2 + y^2 + 6x - 4y - 12 = 0$

$$x^2 + 6x + y^2 - 4y = 12$$

$$(x+3)^2 + (y-2)^2 = 25$$

$$|z + 3 - 2i| = 5$$

$$[(z + 3 - 2i)(\bar{z} + 3 + 2i) = 25]$$

(ii) Find the centre and radius of;

a) $|z - 5 - i| = 2$

centre : $(5,1)$

radius : 2 units

b) $(z + 4 + i)(\bar{z} + 4 - i) = 49$

centre : $(-4,-1)$

radius : 7 units

c) $|3z| = |z+2-i|$

$$3|z| = |z+2-i|$$

$$9x^2 + 9y^2 = (x+2)^2 + (y-1)^2$$

$$9x^2 + 9y^2 = x^2 + 4x + 4 + y^2 - 2y + 1$$

$$8x^2 - 4x + 8y^2 + 2y = 5$$

$$x^2 - \frac{1}{2}x + y^2 + \frac{1}{4}y = \frac{5}{8}$$

$$\left(x - \frac{1}{4}\right)^2 + \left(y + \frac{1}{8}\right)^2 = \frac{45}{64}$$

centre: $\left(\frac{1}{4}, -\frac{1}{8}\right)$

radius: $\frac{3\sqrt{5}}{8}$ units

d) $z\bar{z} + 2(z + \bar{z}) = 0$

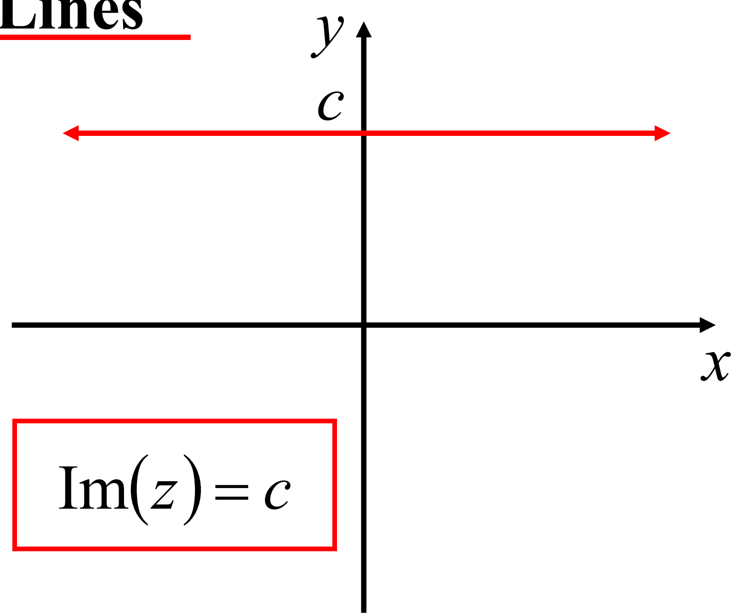
$$x^2 + y^2 + 4x = 0$$

$$(x+2)^2 + y^2 = 4$$

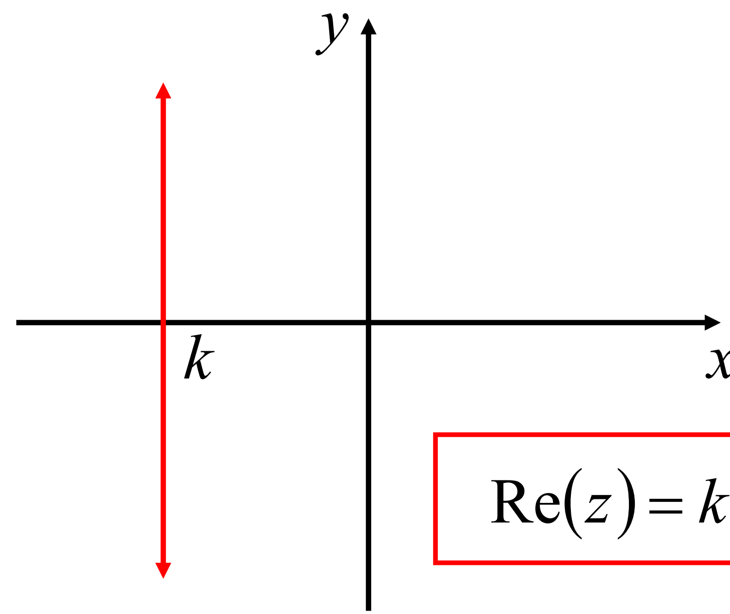
centre: $(-2,0)$

radius: 2 units

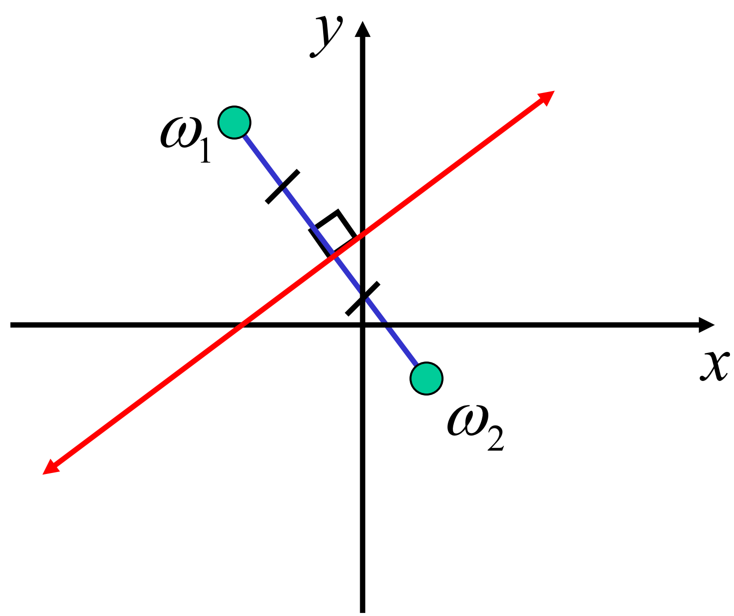
Lines



$$\operatorname{Im}(z) = c$$



$$\operatorname{Re}(z) = k$$



$$|z - \omega_1| = |z - \omega_2|$$

$$e.g. \left|z-1-i\right| = \left|z+2+i\right|$$

$$(x-1)^2+(y-1)^2=(x+2)^2+(y+1)^2$$

$$x^2-2x+1+y^2-2y+1=x^2+4x+4+y^2+2y+1$$

$$\underline{6x+4y+3=0}$$

OR $\perp$ bisector of $(1,1)$ and $(-2,-1)$

$$M=\left(\frac{1-2}{2},\frac{1-1}{2}\right) \qquad m=\frac{1+1}{1+2}$$

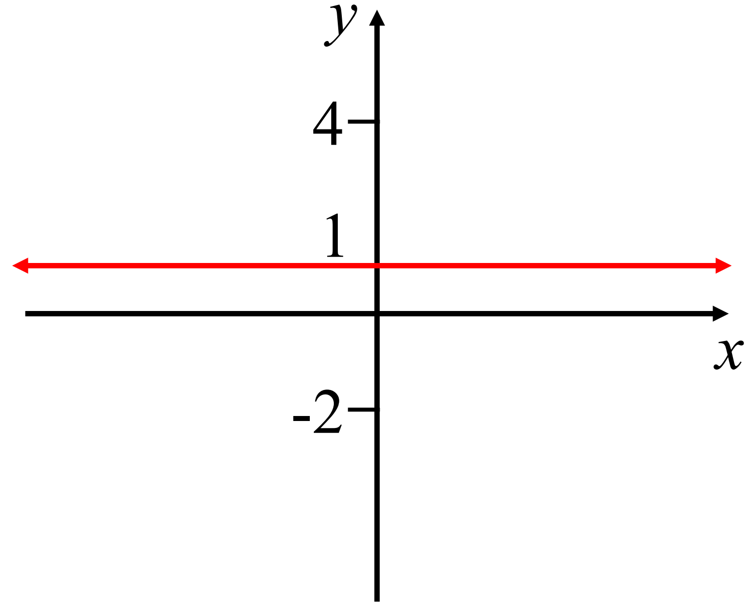
$$=\left(-\frac{1}{2},0\right) \qquad =\frac{2}{3} \qquad \therefore \text{ required slope is } -\frac{3}{2}$$

$$y-0=-\frac{3}{2}\left(x+\frac{1}{2}\right)$$

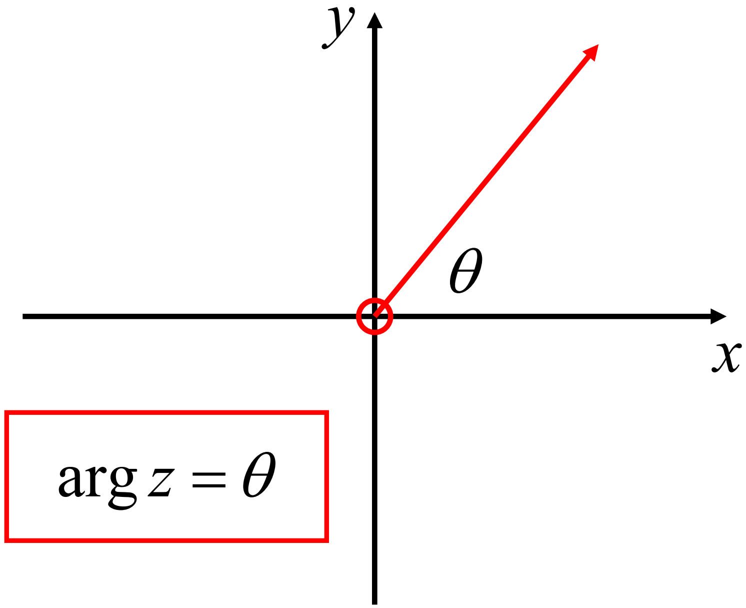
$$2y=-3x-\frac{3}{2}$$

$$\underline{6x+4y+3=0}$$

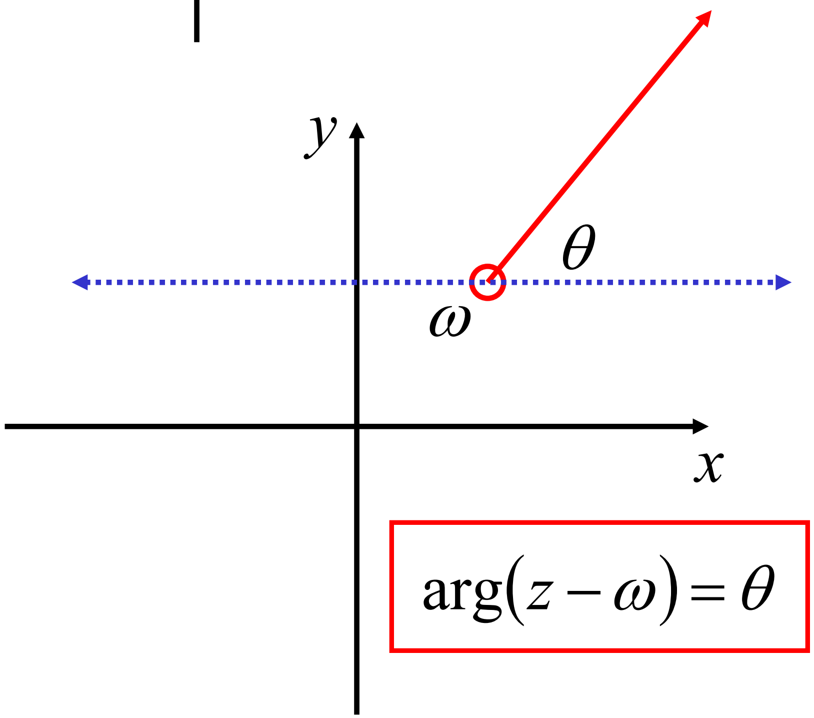
(ii) Sketch $|z+2i| = |z-4i|$



Rays

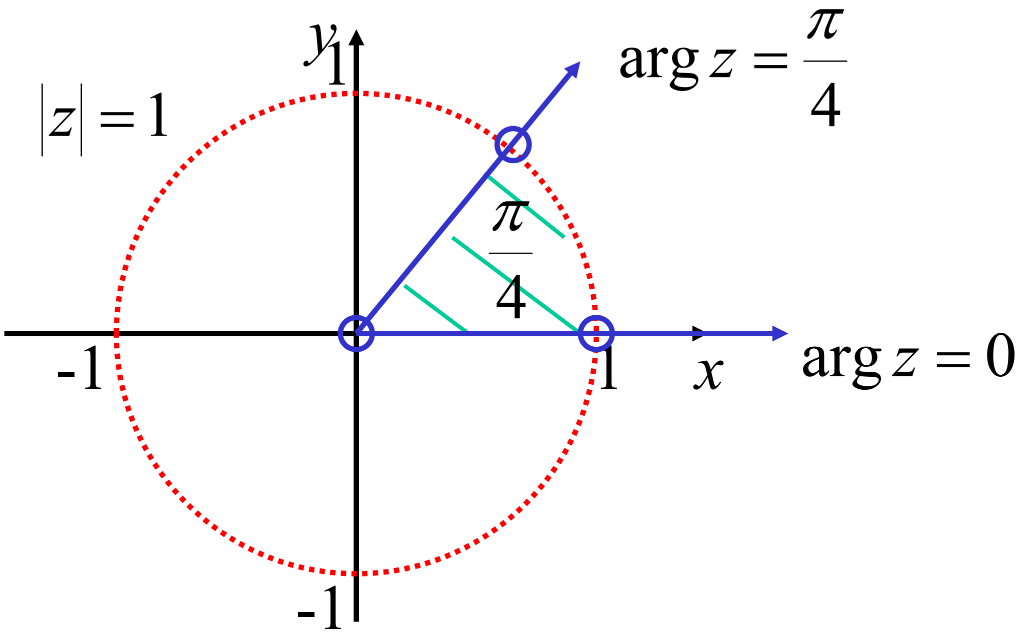


$\arg z = \theta$

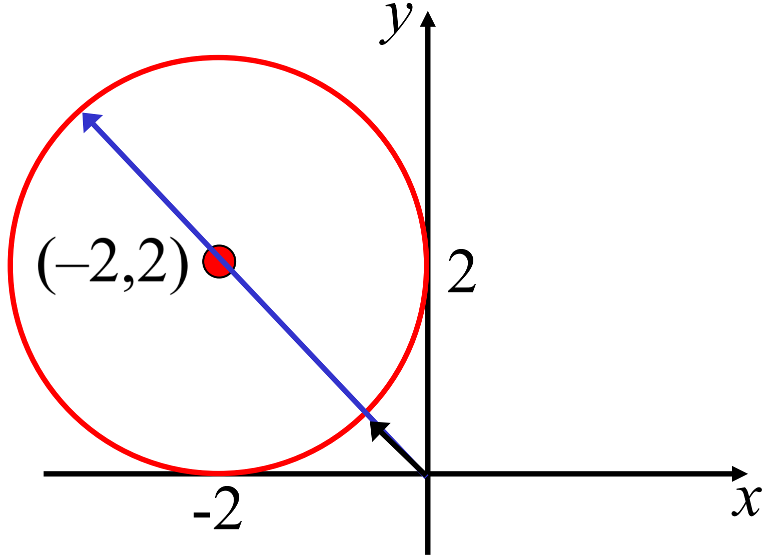


$\arg(z-\omega) = \theta$

e.g. $|z| < 1$ and $0 \le \arg z \le \frac{\pi}{4}$



e.g. (i) On an Argand diagram, sketch $|z + 2 - 2i| = 2$



(ii) Find the possible values of arg z

The tangents drawn from the origin to the circle will create the vectors with the minimum and maximum arguments

$$\frac{\pi}{2} \leq \arg z \leq \pi$$

(iii) Find the minimum and maximum values of $|z|$

Draw a secant that goes through the origin and the centre of the circle, this will create the vectors with minimum and maximum modulus.

distance to centre = $2\sqrt{2}$

radius = 2

min $|z| = 2\sqrt{2} - 2$

max $|z| = 2\sqrt{2} + 2$

e.g. **2021 Extension 2 Question 16c)**

Sketch the region of the complex plane defined by $\mathrm{Re}(z) \geq \mathrm{Arg}(z)$ where $\mathrm{Arg}(z)$ is the principal argument of z.

$$Re(z) \geq \arg(z) \quad , \quad -\pi < \arg(z) \leq \pi$$

boundary curve:

$$x = \tan^{-1}\left(\frac{y}{x}\right)$$

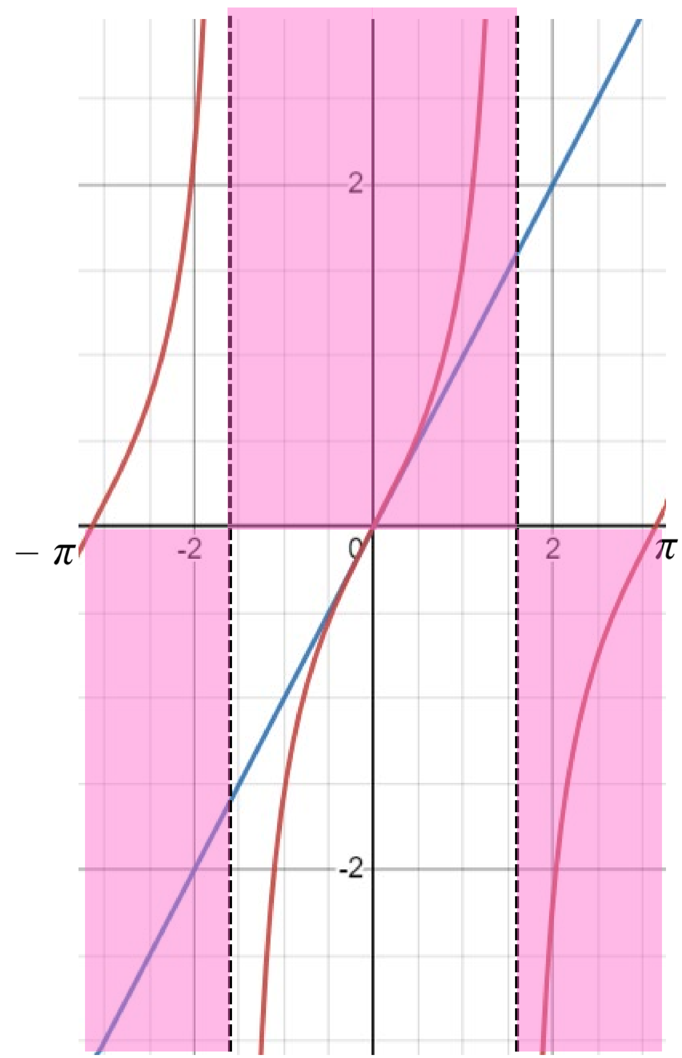
$$\tan x = \frac{y}{x}$$

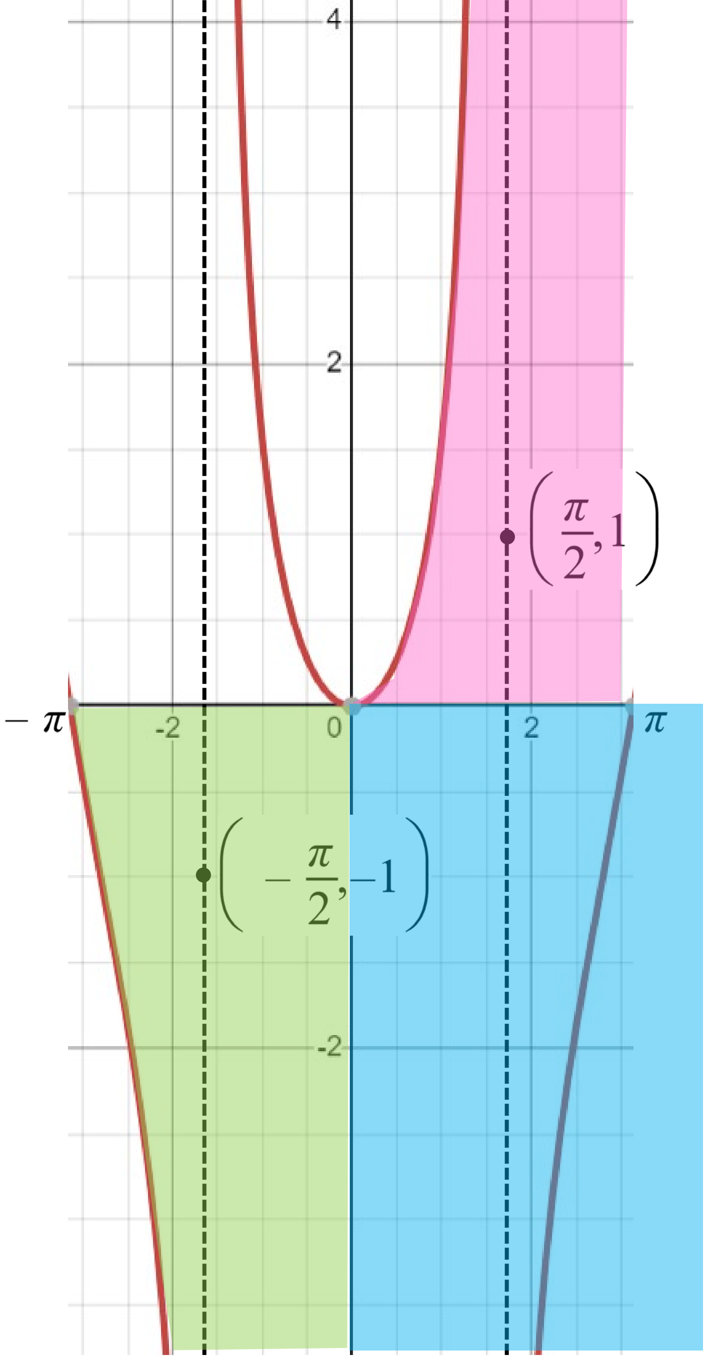
$$y = x\tan x$$

$y = x$ is odd function

$y = \tan x$ is odd function

$\therefore y = x\tan x$ is even function





testing regions:

Q1: $0 < \arg z < \frac{\pi}{2}$ test $\left(\frac{\pi}{2},1\right) = \frac{\pi}{2}+i$

$\frac{\pi}{2} \geq \arg\left(\frac{\pi}{2}+i\right)$ ✓

Q2 : $\frac{\pi}{2} < \arg z < \pi$ all points have $\text{Re}(z) < 0$, so no points are included

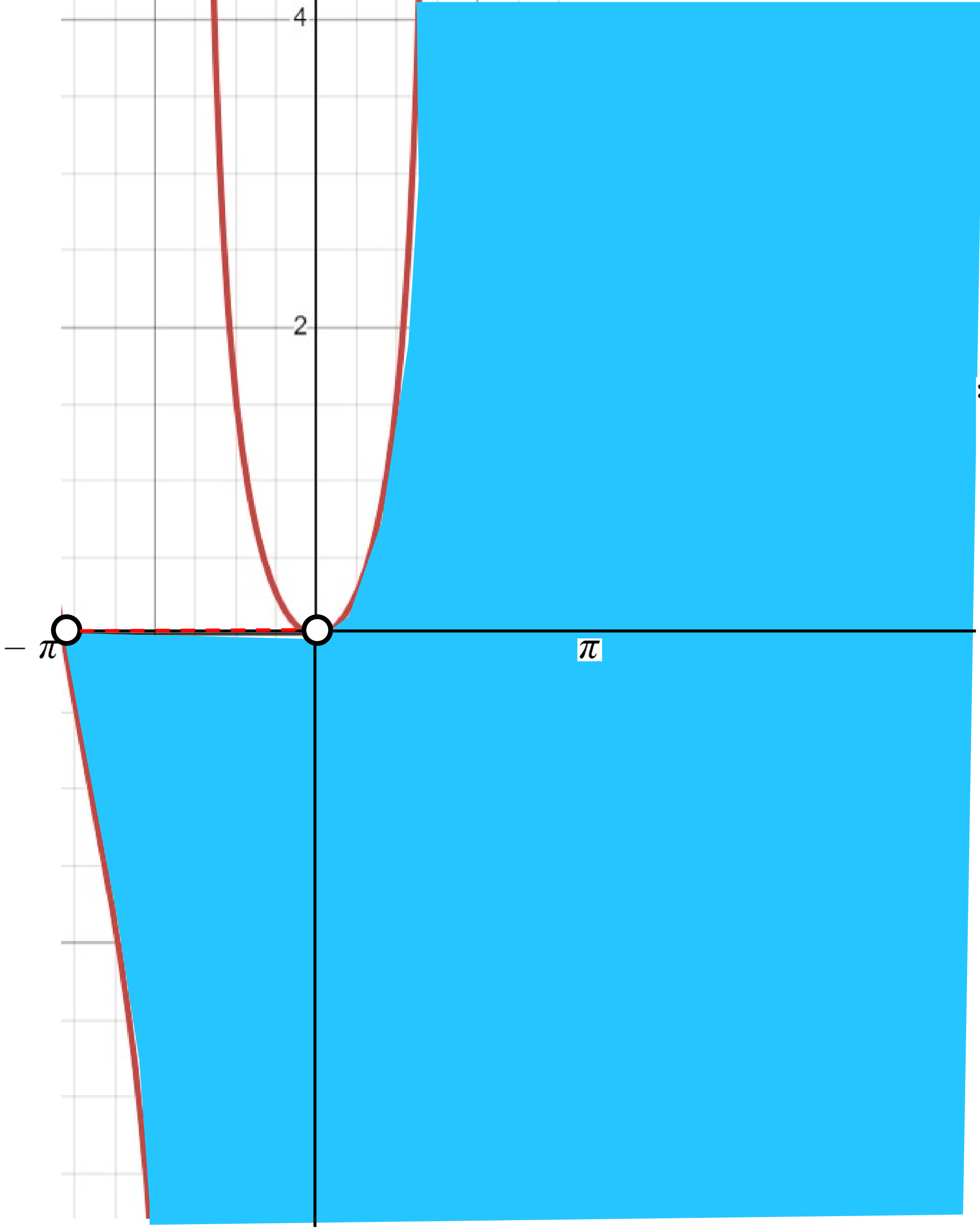
Q3 : $-\pi < \arg z < -\frac{\pi}{2}$

test $\left(-\frac{\pi}{2},-1\right) = -\frac{\pi}{2} - i$

$-\frac{\pi}{2} \geq \arg\left(-\frac{\pi}{2} - i\right)$ ✓

Q4 : $-\frac{\pi}{2} < \arg z < 0$

all points have $\text{Re}(z) > 0$, so all points are included



some things to note about the solution:

* arg(0) is undefined so (0,0) is not part of the solution

* the interval from (0,0) to $(-\pi, 0)$ is also not included as $\arg(z)$ is defined to be π at these points

* the region does not end at $x = \pi$ as it is $-\pi < \arg(z) \leq \pi$ NOT $-\pi < z \leq \pi$

Exercise 1F;
1 to 5 ace etc, 6, 7 ac, 10, 11, 12a, 13

Patel: **Exercise 4M;**
1ac, 2bd, 3ac, 4bdf, 5bd, 6ac

Patel: **Exercise 4N;**
1acfhj, 2ace, 3acegikl, 4ace