Concavity

The second deriviative measures the change in slope with respect to *x*, this is known as **concavity**

If $f''(x) > 0$, the curve is concave up

If $f''(x) < 0$, the curve is concave down

If $f''(x) = 0$, possible point of inflection

e.g. By looking at the second derivative sketch $y = x^3 + 5x^2 + 3x + 2$

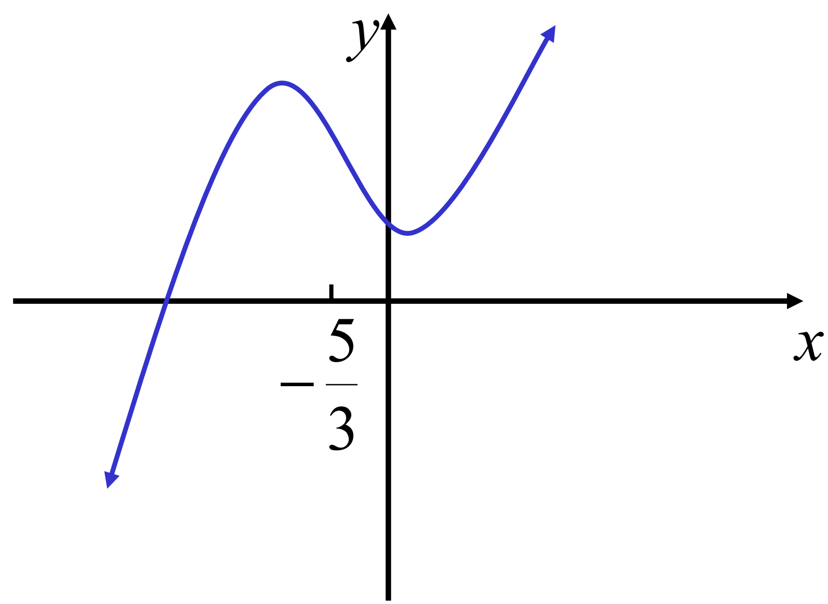
$$\frac{dy}{dx} = 3x^2 + 10x + 3$$

$$\frac{d^2y}{dx^2} = 6x + 10$$

Curve is concave up when $\frac{d^2y}{dx^2} > 0$

i.e. $6x + 10 > 0$

$$x > -\frac{5}{3}$$



Turning Points

All turning points are stationary points.

If $f''(x) > 0$, minimum turning point

If $f''(x) < 0$, maximum turning point

e.g. Find the turning points of $y = x^3 + x^2 - x + 1$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

$$\frac{d^2y}{dx^2} = 6x + 2$$

Stationary points occur when $\frac{dy}{dx} = 0$

i.e. $3x^2 + 2x - 1 = 0$

$$(3x - 1)(x + 1) = 0$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -1$$

$$\text{when } x = -1, \frac{d^2y}{dx^2} = 6(-1) + 2$$

$$= -4 < 0$$

$$\therefore (\text{-}1,2) \text{ is a maximum turning point}$$

$$\text{when } x = \frac{1}{3}, \frac{d^2y}{dx^2} = 6\left(\frac{1}{3}\right) + 2$$

$$= 4 > 0$$

$$\therefore \left(\frac{1}{3}, \frac{22}{27}\right) \text{ is a minimum turning point}$$

Inflection Points

A point of inflection is where there is a **change in concavity**, to see if there is a change, check either side of the point.

e.g. Find the inflection point(s) of $y = 4x^3 + 6x^2 + 2$

$$\frac{dy}{dx} = 12x^2 + 12x \qquad \frac{d^2y}{dx^2} = 24x + 12$$

Possible points of inflection occur when $\frac{d^2y}{dx^2} = 0$

$i.e.\ \ 24x + 12 = 0$

$$x = -\frac{1}{2}$$

x	$-\frac{1}{2}^{-}$ (-1)	$-\frac{1}{2}$	$-\frac{1}{2}^{+}$ (0)
$\frac{d^2y}{dx^2}$	(-12)	0	(12)

$\therefore$ there is a change in concavity

$\therefore \left(-\frac{1}{2}, 3\right)$ is a point of inflection

Horizontal Point of Inflection; $\frac{dy}{dx} = 0 \quad \frac{d^2y}{dx^2} = 0 \quad \frac{d^3y}{dx^3} \neq 0$

Alternative Way of Finding Inflection Points

Possible points of inflection occur when $\frac{d^2y}{dx^2} = 0$

If the first non-zero derivative is of odd order,

i.e $\frac{d^3y}{dx^3} \neq 0$ or $\frac{d^5y}{dx^5} \neq 0$ or $\frac{d^7y}{dx^7} \neq 0$ etc

then it is a point of inflection

If the first non-zero derivative is of even order,

i.e $\frac{d^4y}{dx^4} \neq 0$ or $\frac{d^6y}{dx^6} \neq 0$ or $\frac{d^8y}{dx^8} \neq 0$ etc

then it is not a point of inflection

$$e.g.\ \frac{d^3y}{dx^3} = 24$$

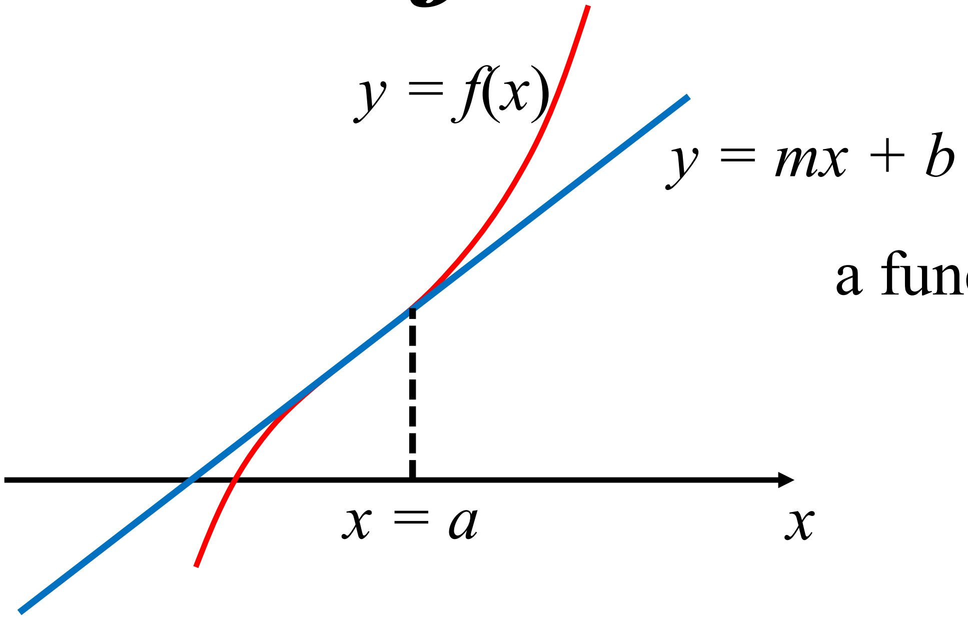
$$\text{when } x = -\frac{1}{2}, \frac{d^3y}{dx^3} = 24 \neq 0$$

$$\therefore \text{there is a change in concavity}$$

$$\therefore \left(-\frac{1}{2}, 3\right) \text{is a point of inflection}$$

Exercise 4E; 1, 2bc, 3bd, 4a, 6, 7, 9, 11, 12ad, 13, 15ad, 16, 17, 19 to 23

Inflection Points



a function inflects through its tangent at that point

let's investigate a new function

$$g(x) = f(x) - (mx + b)$$

$$g(x) = f(x) - (mx + b)$$

$$g'(x) = f'(x) - m$$

$$g''(x) = f''(x)$$

$$g'''(x) = f'''(x)$$

$$\vdots$$

$$g^n(x) = f^n(x)$$

Now

$$\begin{aligned} g(a) &= f(a) - (ma + b) \\ &= f(a) - f(a) \\ &= 0 \end{aligned}$$

i.e. $x = a$ is a root of $g(x)$

$$\begin{aligned} g'(a) &= f'(a) - m \\ &= m - m \\ &= 0 \end{aligned}$$

i.e. $x = a$ is also a stationary point of $g(x)$

we also know that $x = a$ is an inflection point of $f(x)$

$$\therefore g''(a) = f''(a) = 0$$

x	a^-	a	a^+
$g''(x) = f''(x)$	< 0	0	> 0

there is a change in concavity

$x = a$ is a horizontal point of inflection of $g(x)$

All differentiable functions can be represented by a Taylor series (polynomial of infinite order)

If $x = a$ is a horizontal point of inflection of $g(x)$ at this root, then $g(x)$ can be represented by a Taylor series with a factor;

$$(x - a)^{2k+1}, \quad k \in \mathbb{Z}^+$$

$$\text{i.e. } g(x) = (x - a)^{2k+1} Q(x)$$

($x = a$ is an odd-ordered root)

$$g'(a) = g''(a) = \ldots = g^{2k}(a) = 0$$

$$g^{2k+1}(a) \neq 0$$

otherwise it would be a root of order greater than $2k + 1$

i.e. the first non-zero derivative is the same order as the root but

$$f''(a) = g''(a) = 0$$

$$f'''(a) = g'''(a) = 0$$

$$\vdots$$

$$f^{2k}(a) = g^{2k}(a) = 0$$

$$f^{2k+1}(a) = g^{2k+1}(x) \neq 0$$

i.e. if $x = a$ is an inflection point of $f(x)$, then the first non-zero derivative is

$$f^{2k+1}(x)$$

a derivative of odd order