Deductive Reasoning

An argument is valid iff it takes a form that makes it impossible for the premises to be true and the conclusion nevertheless false.

(1) Direct Proof (*modus ponens)*

$$(P \land (P \Rightarrow Q)) \Rightarrow Q$$

P	Q	$(P$	$\land$	$(P$	$\Rightarrow$	$Q))$	$\Rightarrow$	Q
T	T	T	T	T	T	T	T	T
T	F	T	F	T	F	F	T	F
F	T	F	F	F	T	T	T	T
F	F	F	F	F	T	F	T	F

$$((P \Rightarrow Q) \land (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$$

e.g. (i) Prove that if a number is odd, then its square is also odd

Let n be an odd integer

$n = 2k + 1 \quad , \quad k \in \mathbb{Z}$

$n^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2P + 1$ where $P = (2k^2 + 2k) \in \mathbb{Z}$

hence if n is odd, n^2 is also odd

(ii) **2021 Extension 2 HSC Question 15 d)**

Prove that $2^n + 3^n \neq 5^n$ for all integers $n \geq 2$

$$5^n = (2+3)^n$$

$$= \binom{n}{0} 2^n + \binom{n}{1} 2^{n-1} 3 + \binom{n}{2} 2^{n-2} 3^2 + \ldots + \binom{n}{n} 3^n$$

$$> 2^n + 3^n \quad \forall \; n \in \mathbb{Z} : n \geq 2$$

thus $5^n \neq 2^n + 3^n$ for $n \geq 2$

(iii) Prove that the sum of the squares of five consecutive integers is divisible by 5

If p , $q \in \mathbb{Z}$ and q is divisible by p then $\exists\ n \in \mathbb{Z} : q = pn$

Let $q = (n-2)^2 + (n-1)^2 + n^2 + (n+1)^2 + (n+2)^2 \quad , \quad n \in \mathbb{Z}$

$= 5n^2 + 4 + 1 + 1 + 4$

$= 5n^2 + 10$

$= 5(n^2 + 2)$

$= 5P \quad$ where $P = (n^2 + 2) \in \mathbb{Z}$

hence the sum of the squares of five consecutive integers is divisible by 5

(2) Proof by Contraposition (*modus tollens*)

$$(\neg Q \Rightarrow \neg P) \Leftrightarrow (P \Rightarrow Q)$$

e.g. Prove that if $2^n - 1$, $n \in \mathbb{N}$, is prime then n is prime

Let $n = pq$, $p , q \in \mathbb{N}$ and $p , q \neq 1$ *(i.e. n is not prime)*

$$2^n - 1 = 2^{pq} - 1$$

$$= (2^p)^q - 1$$

$$= (2^p - 1)\left(1 + 2^p + 2^{2p} + \ldots + 2^{(q-1)p}\right)$$

$$= PQ \text{ , where } P = (2^p - 1) \neq 1$$

$$\text{and } Q = \left(1 + 2^p + 2^{2p} + \ldots + 2^{(q-1)p}\right) \neq 1 \ \forall \ P , Q \in \mathbb{N}$$

$\therefore$ if n is not prime, then $2^n - 1$ is not prime

hence if $2^n - 1$ is prime then n is prime, by contraposition

(ii) **2022 Extension 2 HSC Question 13 a)**

Prove that for all integers n with $n \geq 3$, if $2^n - 1$ is prime, then n cannot be even.

Let n be an even number ≥ 3

i.e. $n = 2k$ $\qquad\qquad\qquad\qquad$ $k \in \mathbb{Z} : k \geq 2$

$$2^n - 1 = 2^{2k} - 1$$
$$= (2^k - 1)(2^k + 1)$$

Now $2^k + 1 > 2^k - 1$

and $2^k - 1 \geq 2^4 - 1$

$= 3$ $\qquad\qquad\qquad\qquad$ $(k \geq 2)$

thus $2^n - 1$ has two different factors, neither of which is 1

$\therefore$ if n is an even number then $2^n - 1$ is not prime

hence if $2^n - 1$ is prime then n is not even, by contraposition

(3) Proof by Contradiction (*reductio ad impossible – indirect proof*)

$$(\neg(P \Rightarrow Q) \Rightarrow (R \wedge \neg R)) \Rightarrow (P \Rightarrow Q)$$

P	Q	$(\neg(P \Rightarrow Q) \Rightarrow (R \wedge \neg R)) \Rightarrow (P \Rightarrow Q)$						
T	**T**	**F**	**T**	**F**	**T**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**	**T**	**T**
F	**F**	**F**	**T**	**F**	**T**	**F**	**T**	**F**

e.g. (i) Prove $\log_2 5$ is irrational

Assume $\log_2 5$ is rational

i.e. $\log_2 5 = \dfrac{p}{q}$, where p and q are coprime

$$2^{\frac{p}{q}} = 5$$

$$2^p = 5^q$$

so LHS is even and RHS is odd, which is a contradiction

$\therefore \log_2 5$ is irrational

(ii) Prove that there are no integers a and b such that $18a + 6b = 1$

Assume $\exists\ a, b \in \mathbb{Z} : 18a + 6b = 1$

$$18a + 6b = 1$$
$$6(3a + b) = 1$$
$$3a + b = \frac{1}{6}$$

however $3a + b \in \mathbb{Z}$

$$3a + b \neq \frac{1}{6}, \text{ which is a contradiction}$$

Thus there are no integers a and b such that $18a + 6b = 1$

e.g. **2020 Extension 2 HSC Question 15**

In the set of integers, let P be the proposition:

"If $k + 1$ is divisible by 3, then $k^3 + 1$ is divisible by 3"

(i) Prove that the proposition is true

$$\text{Let } k + 1 = 3P \ , \ \text{where } P \in \mathbb{Z} \ \forall \ k \in \mathbb{Z}$$

$$\begin{aligned} k^3 + 1 &= (k + 1)(k^2 - k + 1) \\ &= 3P(k^2 - k + 1) \\ &= 3Q \quad \text{where } Q = P(k^2 - k + 1) \in \mathbb{Z} \end{aligned}$$

Thus if $k + 1$ is divisible by 3, then $k^3 + 1$ is divisible by 3

(ii) Write down the contrapositive of the proposition P

If $k^3 + 1$ is not divisible by 3 then $k + 1$ is not divisible by 3,

(iii) Write down the converse of the proposition P and state, with reasons, whether this converse is true or false

If $k^3 + 1$ is divisible by 3 then $k + 1$ is divisible by 3

Exercise 2B; 1b, 2c, 3a, 4a, 7, 10, 12, 16, 17b

Exercise 2C; 1, 3, 4, 5, 7, 9, 11, 13

$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$$

P : $k^3 + 1$ is divisible by 3 Q : $k + 1$ is divisible by 3

$$\begin{aligned} k^3 + 1 &= (k+1)(k^2 - k + 1) \\ &= (k+1)(k^2 + 2k + 1 - 3k) \\ &= (k+1)[(k+1)^2 - 3k] \end{aligned}$$

If $k + 1$ is not divisible by 3 then neither is $(k + 1)^2$

Thus $[(k + 1)^2 - 3k]$ is not divisible by 3, which means

$(k + 1)[(k + 1)^2 - 3k]$ is not divisible by 3

i.e. If $k + 1$ is not divisible by 3 then $k^3 + 1$ is not divisible by 3

$\therefore$ If $k^3 + 1$ is divisible by 3 then $k + 1$ is divisible by 3 by contraposition