Algebraic Inequalities

Properties of Inequalities

If $a > b$ then $a \pm x > b \pm x$	you can add or subtract the same amount to both sides
If $a > b$ then $ax > bx$ for $x > 0$	you can multiply or divide both sides by a positive amount
If $a > b$ then $ax < bx$ for $x < 0$	the inequality sign changes when you multiply or divide both sides by a negative amount
If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$	when both sides are positive, the inequality sign changes when you take the reciprocal of both sides
If $a > b > 0$ then $a^2 > b^2$	when both sides are positive, you can square both sides
If $a > b$ and $b > c$ then $a > c$	inequalities of the same sign can be linked
If $a > b > 0$ and $c > d > 0$ then $ac > bd$	when both sides are positive, inequalities of the same sign can be multiplied
If $a > b > 0$ and $c > d > 0$ then $a + c > b + d$	when both sides are positive, inequalities of the same sign can be added

Inequality Techniques

To prove $x \geq y$, it can be easier to prove $x - y \geq 0$

e.g. (i) (1995) Prove $pq \leq \frac{p^2+q^2}{2}$

$$\frac{p^2+q^2}{2} - pq = \frac{p^2-2pq+q^2}{2}$$

$$= \frac{(p-q)^2}{2}$$

$$\geq 0$$

$$\therefore \frac{p^2+q^2}{2} \geq pq$$

OR Assume $pq > \frac{p^2+q^2}{2}$

$$2pq > p^2+q^2$$

$$0 > p^2-2pq+q^2$$

$$0 > (p-q)^2$$

But $(p-q)^2 > 0$ which is a contradiction

$$\therefore pq \leq \frac{p^2+q^2}{2}$$

Start with a known result

$(ii)\,(1994)$ a) Prove $a^2+b^2+c^2>ab+bc+ac$

$$(a-b)^2>0$$
$$a^2-2ab+b^2>0$$
$$\therefore a^2+b^2>2ab$$
$$a^2+c^2>2ac$$
$$b^2+c^2>2bc$$
$$2a^2+2b^2+2c^2>2ab+2ac+2bc$$
$$\underline{a^2+b^2+c^2>ab+ac+bc}$$

b) If $a+b+c=1$, prove $ab+ac+bc<\frac{1}{3}$

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)$$
$$\therefore (a+b+c)^2-2(ab+ac+bc)>ab+ac+bc$$
$$3(ab+ac+bc)<(a+b+c)^2$$
$$3(ab+ac+bc)<1$$
$$\underline{ab+ac+bc<\frac{1}{3}}$$

Substitute different expressions into known inequalities

c) Prove $\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$

$$a^2+b^2+c^2 \geq ab+ac+bc$$

$$a^2+b^2+c^2-ab-ac-bc \geq 0$$

$$(a+b+c)(a^2+b^2+c^2-ab-ac-bc) \geq 0$$

$$a^3+ab^2+ac^2-a^2b-a^2c-abc+a^2b+b^3+bc^2-ab^2-abc-b^2c$$

$$+a^2c+b^2c+c^3-abc-ac^2-bc^2 \geq 0$$

$$a^3+b^3+c^3-3abc \geq 0$$

$$\frac{1}{3}(a^3+b^3+c^3) \geq abc$$

let $a=a^{\frac{1}{3}}, b=b^{\frac{1}{3}}, c=c^{\frac{1}{3}}$

$$\frac{1}{3}(a+b+c) \geq a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$$

$$\frac{1}{3}(a+b+c) \geq \sqrt[3]{abc}$$

Arithmetic Mean $\geq$ Geometric Mean

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

d) Suppose $(1+x)(1+y)(1+z) = 8$, prove $xyz \leq 1$

$$(1+x)(1+y)(1+z) = 8$$

$$1 + x + y + xy + z + xz + yz + xyz = 8$$

$$\frac{1}{3}(x+y+z) \geq \sqrt[3]{xyz} \qquad \text{AM} \geq \text{GM}$$

$$x + y + z \geq 3\sqrt[3]{xyz}$$

$$xy + yz + xz \geq 3\sqrt[3]{(xy)(yz)(xz)}$$

$$xy + yz + xz \geq 3\sqrt[3]{x^2 y^2 z^2}$$

$$xy + yz + xz \geq 3\left(\sqrt[3]{xyz}\right)^2$$

$$1+x+y+z+xy+xz+yz+xyz=8$$

$$1+3\sqrt[3]{xyz}+3\left(\sqrt[3]{xyz}\right)^2+xyz\le 8$$

$$1+3\sqrt[3]{xyz}+3\left(\sqrt[3]{xyz}\right)^2+\left(\sqrt[3]{xyz}\right)^3\le 8$$

$$\left(1+\sqrt[3]{xyz}\right)^3\le 8$$

$$1+\sqrt[3]{xyz}\le 2$$

$$\sqrt[3]{xyz}\le 1$$

$$xyz\le 1$$

OR

AM $\geq$ GM

$$(1+x) \geq 2\sqrt{x}$$

$$(1+y) \geq 2\sqrt{y}$$

$$(1+z) \geq 2\sqrt{z}$$

$$(1+x)(1+y)(1+z) \geq 2\sqrt{x} \times 2\sqrt{y} \times 2\sqrt{z}$$

$$= 8\sqrt{xyz}$$

$$\therefore 8 \geq 8\sqrt{xyz}$$

$$1 \geq 1\sqrt{xyz}$$

$$\sqrt{xyz} \leq 1$$

$$xyz \leq 1$$

(iii) Prove $\dfrac{9}{a+b+c} \leq \dfrac{2}{a+b} + \dfrac{2}{b+c} + \dfrac{2}{a+c} \leq \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 2\sqrt{ab} \times 2\sqrt{\frac{1}{ab}} = 4$$

$$\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$$

$$\frac{1}{b}+\frac{1}{c} \geq \frac{4}{b+c}$$

$$\frac{1}{a}+\frac{1}{c} \geq \frac{4}{a+c}$$

$$\frac{2}{a}+\frac{2}{b}+\frac{2}{c} \geq \frac{4}{a+b}+\frac{4}{b+c}+\frac{4}{a+c}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c}$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3\sqrt[3]{abc} \times 3\sqrt[3]{\frac{1}{abc}} = 9$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{9}{a+b+c}$$

Replace a with $a + b$

b with $b + c$

c with $a + c$

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{a+c} \geq \frac{9}{2(a+b+c)}$$

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{a+c} \geq \frac{9}{a+b+c}$$

$$\frac{9}{a+b+c} \leq \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{a+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

(*iv*) a) If $a > 0$ is a real number, show that $a + \frac{1}{a} \geq 2$

$$a + \frac{1}{a} \geq 2\sqrt{a \times \frac{1}{a}}$$ AM ≥ GM ***OR*** $$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 \geq 0$$

$$a + \frac{1}{a} \geq 2\sqrt{1}$$ $$a - 2 + \frac{1}{a} \geq 0$$

$$a + \frac{1}{a} \geq 2$$ $$a + \frac{1}{a} \geq 2$$

If $a > 0$, $b > 0$ and $c > 0$;

b) Show that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$

$$\frac{b}{a} + \frac{a}{b} \geq 2$$

$$\frac{c}{a} + \frac{a}{c} \geq 2$$

$$\frac{c}{b} + \frac{b}{c} \geq 2$$

$$\frac{b}{a} + \frac{a}{b} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{c} \geq 6$$

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

c) Show that $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$$

Replace a with $b+c$

b with $a+c$

c with $a+b$

$$\frac{a+c+a+b}{b+c}+\frac{a+b+b+c}{a+c}+\frac{b+c+a+c}{a+b}\geq 6$$

$$\frac{2a}{b+c}+1+\frac{2b}{a+c}+1+\frac{2c}{a+b}+1\geq 6$$

$$\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}\geq 3$$

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$$

(v) **2021 Extension 2 HSC Question 15 a)**

For all non-negative real numbers x and y , $\sqrt{xy} \le \dfrac{x+y}{2}$ (Do NOT prove this)

a) Using this fact, show that for all non-negative real numbers a, b and c,

$$\sqrt{abc} \le \frac{a^2+b^2+2c}{4}$$

$$\frac{a^2+b^2}{4} \ge 2\sqrt{\frac{a^2b^2}{16}}$$

$$= \sqrt{\frac{a^2b^2}{4}}$$

$$= \frac{ab}{2}$$

$$\left(\begin{array}{c} \sqrt{xy} \le \dfrac{x+y}{2} \\ x+y \ge 2\sqrt{xy} \end{array}\right)$$

$$\frac{a^2+b^2+2c}{4} = \frac{a^2+b^2}{4} + \frac{c}{2}$$

$$\ge \frac{ab}{2} + \frac{c}{2}$$

$$\ge 2\sqrt{\frac{abc}{4}}$$

$$= \sqrt{abc}$$

$$\text{thus } \sqrt{abc} \le \frac{a^2+b^2+2c}{4}$$

b) Using part a), or otherwise, show that for all non-negative real numbers *a, b* and *c*

$$\sqrt{abc} \le \frac{a^2+b^2+c^2+a+b+c}{6}$$

$$\sqrt{abc} \le \frac{a^2+b^2+2c}{4}$$

similarly

$$\sqrt{abc} \le \frac{a^2+c^2+2b}{4}$$

$$\sqrt{abc} \le \frac{b^2+c^2+2a}{4}$$

$$\therefore 3\sqrt{abc} \le \frac{2a^2+2b^2+2c^2+2a+2b+2c}{4}$$

$$\sqrt{abc} \le \frac{a^2+b^2+c^2+a+b+c}{6}$$

(vi) **2022 Extension 2 HSC Question 16 c)**

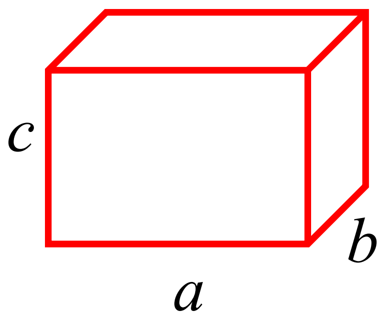
It is given that for positive numbers $x_1, x_2, x_3, \ldots, x_n$ with arithmetic mean A,

$$\frac{x_1 \times x_2 \times x_3 \times \ldots \times x_n}{A^n} \leq 1$$

this is just a confusing way to say $AM > GM$

Suppose that a rectangular prism has dimensions a, b, c and surface area S

(i) Show that $abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$



$$S = 2(ab + ac + bc)$$

$$\frac{ab \times ac \times bc}{\left(\frac{ab + ac + bc}{3}\right)^3} \leq 1$$

$$a^2 b^2 c^2 \leq \left[\frac{2(ab + ac + bc)}{6}\right]^3$$

$$= \left(\frac{S}{6}\right)^3$$

$$\therefore abc \leq \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

(ii) Using part (i) , show that when the rectangular prism with surface area S is a cube, it has a maximum volume

If the solid is a cube then $a = b = c$

$$abc = a^3$$

$$\left(\frac{S}{6}\right)^{\frac{3}{2}} = \left[\frac{2(a^2 + a^2 + a^2)}{6}\right]^{\frac{3}{2}}$$

$$= (a^2)^{\frac{3}{2}}$$

$$= a^3$$

so when $a = b = c$

$$\text{volume} = abc = \left(\frac{S}{6}\right)^{\frac{3}{2}}$$

Exercise 2D; 2b, 4b, 5a, 8, 11, 12, 13, 14, 15, 16, 19, 20

i.e. when the solid is a cube, its volume is at its maximum