Trig Equations

Compound Angle Formulae

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double Angles

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1 \qquad \Rightarrow \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$= 1 - 2\sin^2\theta \qquad \Rightarrow \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angles – *t* results

If $t = \tan\frac{\theta}{2}$;

$$\tan\theta = \frac{2t}{1-t^2} \qquad \sin\theta = \frac{2t}{1+t^2} \qquad \cos\theta = \frac{1-t^2}{1+t^2}$$

Products to Sums

$$2\sin A\cos B = \sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B = \cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

OR $-\cos(A+B)+\cos(A-B)$

Sums to Products*

$$\sin A+\sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos A+\cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos A-\cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

Solving Trig Equations

(1) Get all the angles the same

(2) Match the required answer range to the angle

(3) Get all the trig functions the same

if in doubt, change everything to sin and cos

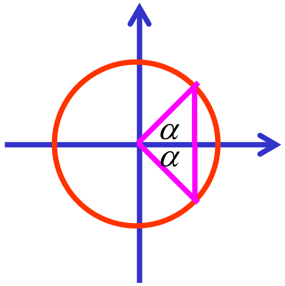
(4) Check your answer(s) solve the original question, NOT just the transformed equation

e.g. (i) $\cos 2\theta = \frac{1}{2}$ $\qquad 0^\circ \le \theta \le 360^\circ$

Q1, Q4 $\qquad 0^\circ \le 2\theta \le 720^\circ$

$\cos\alpha = \frac{1}{2}$

$\alpha = 60^\circ$



$2\theta = \alpha, 360 - \alpha$

$2\theta = 60^\circ, 360 - 60^\circ$

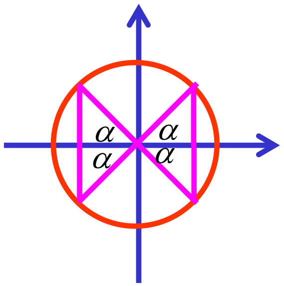
$2\theta = 60^\circ, 300^\circ, 420^\circ, 660^\circ$

$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

OR $\cos 2\theta = \frac{1}{2}$

$0^\circ \le \theta \le 360^\circ$

Q1, Q2, Q3, Q4



$$2\cos^2\theta - 1 = \frac{1}{2}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}$$

$$\cos^2\theta = \frac{3}{4}$$

$$\alpha = 30^\circ$$

$$\theta = \alpha, 180-\alpha, 180+\alpha, 360-\alpha$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

$$\theta = 30^\circ, 180-30^\circ, 180+30^\circ, 360-30^\circ$$

$$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

$$\text{(ii)}\ 4\sec^2 x = 3\tan x + 5 \qquad 0^\circ \le x \le 360^\circ$$

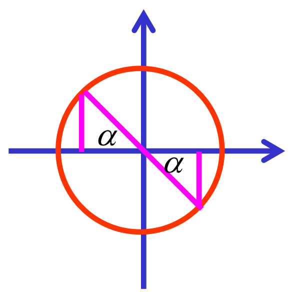
$$4 + 4\tan^2 x = 3\tan x + 5$$

$$4\tan^2 x - 3\tan x - 1 = 0$$

$$(4\tan x + 1)(\tan x - 1) = 0$$

$$\tan x = -\frac{1}{4} \qquad \text{or} \qquad \tan x = 1$$

Q2, Q4



$$\tan\alpha = \frac{1}{4}$$

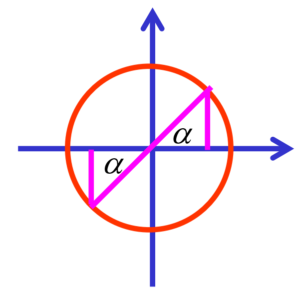
$$\alpha = 14^\circ 2'$$

$$x = 180 - \alpha, 360 - \alpha$$

$$x = 180 - 14^\circ 2', 360 - 14^\circ 2'$$

$$x = 165^\circ 58', 345^\circ 58'$$

Q1, Q3



$$\tan\alpha = 1$$

$$\alpha = 45^\circ$$

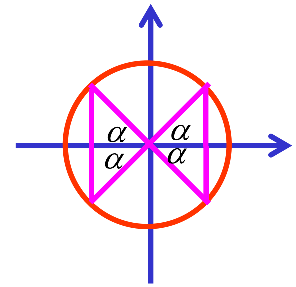
$$x = \alpha, 180 + \alpha$$

$$x = 45^\circ, 180 + 45^\circ$$

$$x = 45^\circ, 225^\circ$$

$$\underline{x = 45^\circ, 165^\circ 58', 225^\circ, 345^\circ 58'}$$

(iii) $\cos 2\theta = 4\cos^2\theta - 2\sin^2\theta$ $\quad 0^\circ \le \theta \le 360^\circ$



$\cos^2\theta - \sin^2\theta = 4\cos^2\theta - 2\sin^2\theta$

$3\cos^2\theta = \sin^2\theta$ $\quad$ Q1, Q2, Q3, Q4

$\tan^2\theta = 3$ $\quad$ $\tan\alpha = \sqrt{3}$

$\tan\theta = \pm\sqrt{3}$ $\quad$ $\alpha = 60^\circ$

$\theta = \alpha, 180-\alpha, 180+\alpha, 360-\alpha$

$\theta = 60^\circ, 180-60^\circ, 180+60^\circ, 360-60^\circ$

$\underline{\theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ}$

(iv) $\cos 2\theta = \sin\theta,$ $\quad 0 \le \theta \le 360^\circ$ $\quad$ 2000 Extension 1 HSC Q2c)

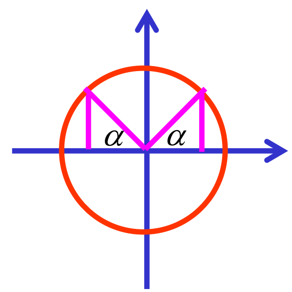
$1 - 2\sin^2\theta = \sin\theta$

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$\sin\theta = \frac{1}{2}$ $\quad$ or $\quad$ $\sin\theta = -1$

Q1, Q2 $\quad$ $\theta = 270^\circ$



$\sin\alpha = \frac{1}{2}$

$\alpha = 30^\circ$

$\theta = \alpha, 180-\alpha$

$\theta = 30^\circ, 180-30^\circ$

$\theta = 30^\circ, 150^\circ$

$\underline{\theta = 30^\circ, 150^\circ, 270^\circ}$

(v) $2\sin^2\theta = \sin 2\theta, \qquad 0 \le \theta \le 360^\circ$ 1992 Extension 1 HSC Q2a)

$$2\sin^2\theta = 2\sin\theta\cos\theta$$

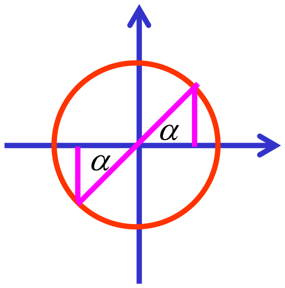
$$2\sin^2\theta - 2\sin\theta\cos\theta = 0$$

$$2\sin\theta(\sin\theta - \cos\theta) = 0$$

$\sin\theta = 0$ or $\sin\theta = \cos\theta$

$\theta = 0^\circ, 180^\circ, 360^\circ$ $\qquad \tan\theta = 1$

Q1, Q3



$$\tan\alpha = 1$$

$$\alpha = 45^\circ$$

$$\theta = \alpha, 180 + \alpha$$

$$\theta = 45^\circ, 180 + 45^\circ$$

$$\theta = 45^\circ, 225^\circ$$

$$\underline{\theta = 0^\circ, 45^\circ, 180^\circ, 225^\circ, 360^\circ}$$

(vi) $$\sin\theta\sin 3\theta = \frac{1}{2} \qquad 0 \le \theta \le 2\pi$$

$$-\cos 4\theta + \cos 2\theta = 1$$

$$-2\cos^2 2\theta + 1 + \cos 2\theta = 1 \qquad 0 \le 2\theta \le 4\pi$$

$$2\cos^2 2\theta - \cos 2\theta = 0$$

$$\cos 2\theta(2\cos 2\theta - 1) = 0$$

$$\cos 2\theta = 0 \qquad \text{or} \qquad \cos 2\theta = \frac{1}{2}$$

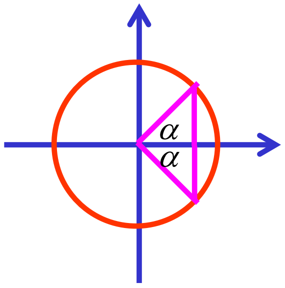
$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Q1, Q4

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{3}$$



$$2\theta = \alpha, 2\pi - \alpha$$

$$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$$

(vii) a) The numbers A, B and C are related by the equations $A = B - d$ and $C = B + d$, where d is a constant.

Show that $\dfrac{\sin A + \sin C}{\cos A + \cos C} = \tan B$

$$\frac{\sin A + \sin C}{\cos A + \cos C} = \frac{\sin(B - d) + \sin(B + d)}{\cos(B - d) + \cos(B + d)}$$

$$\frac{\textit{2 sine half sum cos half diff}}{\textit{2 cos half sum cos half diff}}$$

$$= \frac{2\sin B\cos(-d)}{2\cos B\cos(-d)}$$

(using sum to product)

$$= \frac{\sin B}{\cos B}$$

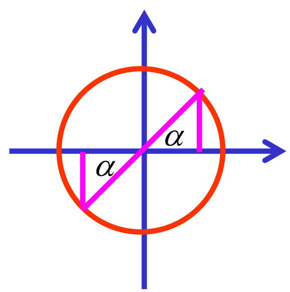
$$= \tan B$$

b) Hence, or otherwise, solve $\dfrac{\sin\frac{5\theta}{7}+\sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7}+\cos\frac{6\theta}{7}}=\sqrt{3}$, for $0\leq\theta\leq 2\pi$

$$\frac{\sin\frac{5\theta}{7}+\sin\frac{6\theta}{7}}{\cos\frac{5\theta}{7}+\cos\frac{6\theta}{7}}=\sqrt{3}$$

$$\tan\frac{11\theta}{14}=\sqrt{3}$$

Q1, Q3



$$\frac{11\theta}{14}=\frac{\pi}{3},\frac{4\pi}{3}$$

$$\theta=\frac{14\pi}{33},\frac{56\pi}{33}$$

$$\begin{aligned} B-d&=A\\ B+d&=C\\ \hline 2B&=A+C\\ B&=\frac{A+C}{2}\end{aligned}$$

$$0\leq\theta\leq 2\pi$$

$$0\leq\frac{11\theta}{14}\leq\frac{11\pi}{7}$$

Exercise 11A; 1, 3, 4ad, 5be 6dij, 9, 10, 11, 12cdf, 13b 14, 15, 16