

Inequalities & Graphs

e.g. (i) Solve $\frac{x^2}{x+2} \leq 1$

$$\frac{x^2}{x+2} = 1$$

$$x^2 = x + 2$$

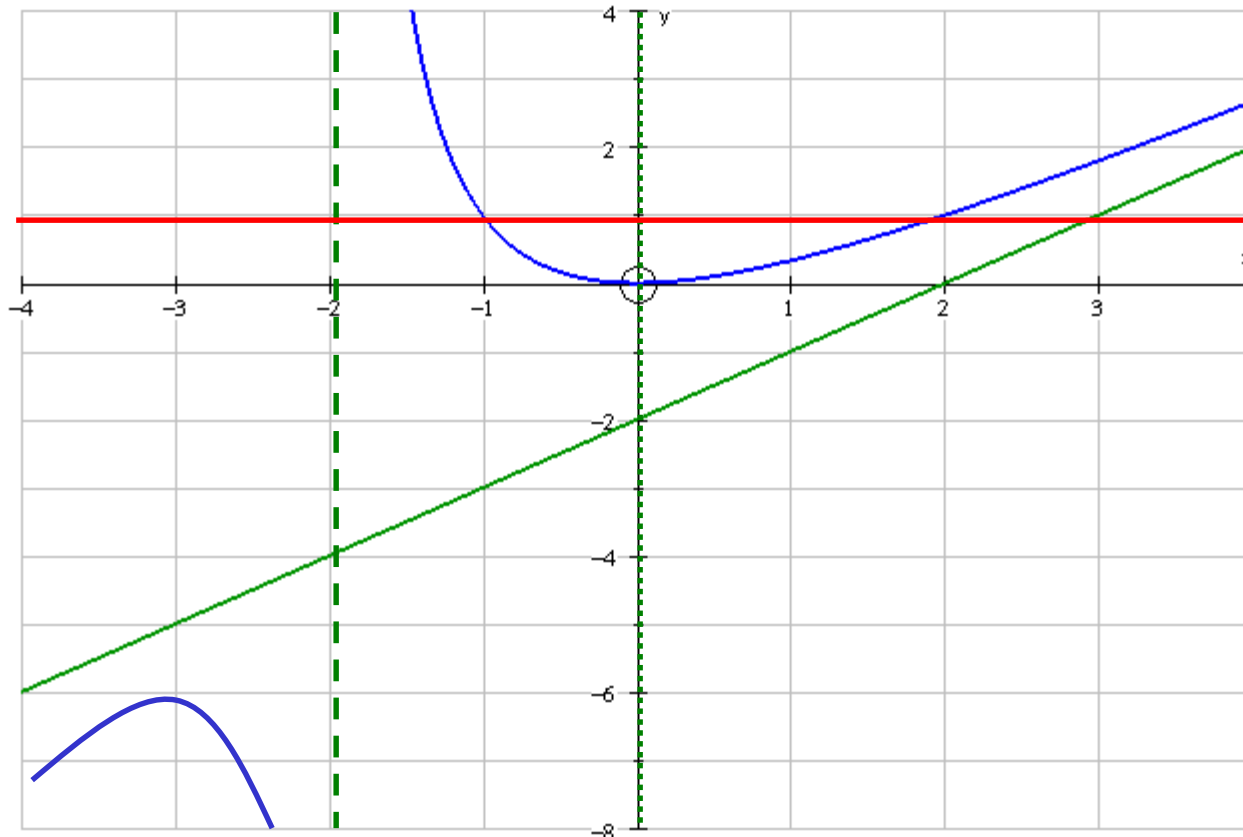
$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ or } x = -1$$

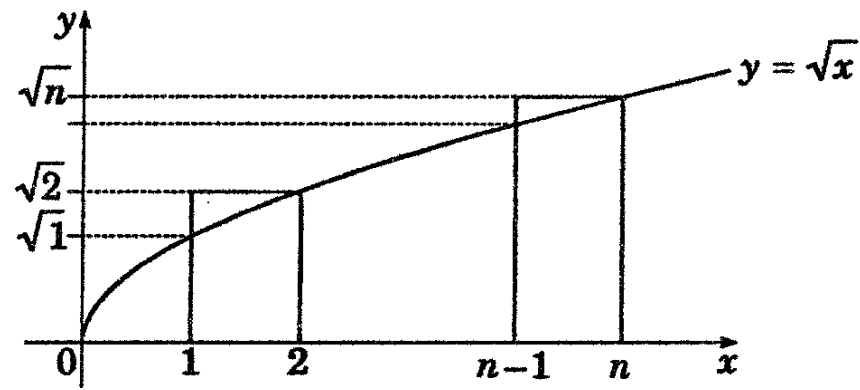
$$\frac{x^2}{x+2} \leq 1$$

$$x < -2 \text{ or } -1 \leq x \leq 2$$



Equation 3: $y = x - 2$
Equation 4: $y = \frac{x^2}{x+2}$

(ii) (1990)



Consider the graph $y = \sqrt{x}$

a) Show that the graph is increasing for all $x \geq 0$

Curve is increasing when $\frac{dy}{dx} > 0$

$$y = \sqrt{x}$$

at $x = 0, y = 0$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

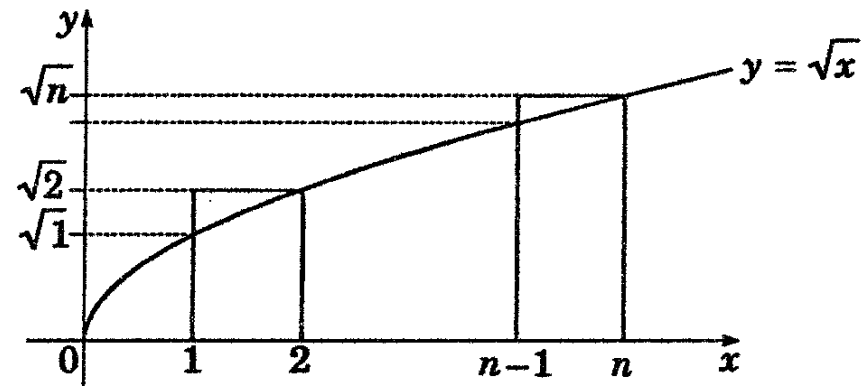
when $x > 0, y > 0$

$\therefore \frac{dy}{dx} > 0$ for $x > 0$

\therefore curve is increasing for $x \geq 0$

b) Hence show that;

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \int_0^n \sqrt{x} dx = \frac{2}{3} n\sqrt{n}$$



As \sqrt{x} is increasing;

Area outer rectangles \geq Area under curve

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \int_0^n \sqrt{x} dx$$

$$= \left[\frac{2}{3} x\sqrt{x} \right]_0^n$$

$$= \frac{2}{3} n\sqrt{n}$$

$$\therefore \sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \geq \int_0^n \sqrt{x} dx = \frac{2}{3} n\sqrt{n}$$

c) Use mathematical induction to show that;

$$\sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \leq \frac{4n+3}{6} \sqrt{n} \text{ for all integers } n \geq 1$$

Prove the result is true for $n = 1$

$$\begin{aligned} L.H.S &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} R.H.S &= \frac{4(1)+3}{6} \sqrt{1} \\ &= \frac{7}{6} \end{aligned}$$

$$\therefore L.H.S \leq R.H.S$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$\text{i.e. } \sqrt{1} + \sqrt{2} + \dots + \sqrt{k} \leq \frac{4k+3}{6} \sqrt{k}$$

Prove the result is true for $n = k + 1$

$$\text{Prove } \sqrt{1} + \sqrt{2} + \dots + \sqrt{k+1} - \frac{4k+7}{6} \sqrt{k+1} \leq 0$$

Proof:

$$\begin{aligned} \sqrt{1} + \sqrt{2} + \dots + \sqrt{k+1} - \frac{4k+7}{6}\sqrt{k+1} &= \sqrt{1} + \sqrt{2} + \dots + \sqrt{k} - \frac{4k+1}{6}\sqrt{k+1} \\ &\leq \frac{4k+3}{6}\sqrt{k} - \frac{4k+1}{6}\sqrt{k+1} \\ &= \frac{\sqrt{(4k+3)^2k} - (4k+1)\sqrt{k+1}}{6} \\ &= \frac{\sqrt{16k^3 + 24k^2 + 9k} - (4k+1)\sqrt{k+1}}{6} \\ &= \frac{\sqrt{(k+1)(16k^2 + 8k + 1)} - 1 - (4k+1)\sqrt{k+1}}{6} \\ &< \frac{\sqrt{(k+1)(16k^2 + 8k + 1)} - (4k+1)\sqrt{k+1}}{6} \\ &= \frac{\sqrt{(k+1)(4k+1)^2} - (4k+1)\sqrt{k+1}}{6} \\ &= \frac{(4k+1)\sqrt{k+1} - (4k+1)\sqrt{k+1}}{6} = 0 \end{aligned}$$

$$\therefore \sqrt{1} + \sqrt{2} + \dots + \sqrt{k+1} - \frac{4k+7}{6}\sqrt{k+1} \leq 0$$

Hence the result is true for $n = k+1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall n \in \mathbb{Z}^+$ by induction

d) Use b) and c) to estimate;

$\sqrt{1} + \sqrt{2} + \dots + \sqrt{10000}$ to the nearest hundred

$$\frac{2}{3}n\sqrt{n} \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{n} \leq \frac{4n+3}{6}\sqrt{n}$$

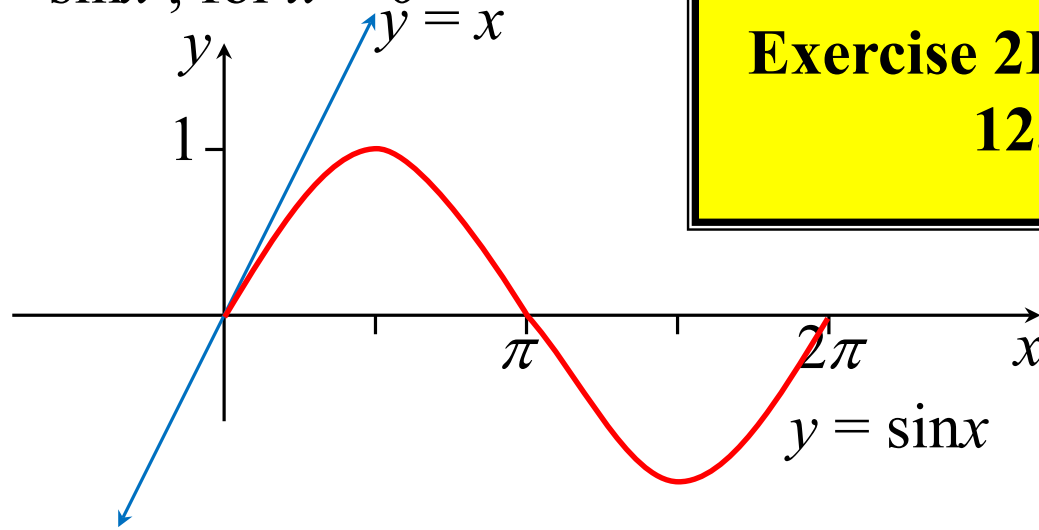
$$\frac{2}{3}(10000)\sqrt{10000} \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} \leq \frac{4(10000)+3}{6}\sqrt{10000}$$

$$666700 \leq \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} \leq 666700$$

$\therefore \sqrt{1} + \sqrt{2} + \dots + \sqrt{10000} = 666700$ to the nearest hundred

(iii) Prove $x > \sin x$, for $x > 0$

**Exercise 2F; 1, 3, 4, 6, 8, 9,
12, 14, 17**



$$f(x) = x$$

$$f'(x) = 1$$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

for $0 < x \leq \frac{\pi}{2}$, $\cos x < 1$

for $x > \frac{\pi}{2}$, $\sin x \leq 1$

$\therefore y = x$ increases faster than $y = \sin x$

$\therefore x > \sin x$, for $x > \frac{\pi}{2}$

$x > \sin x$, for $0 < x \leq \frac{\pi}{2}$

$\therefore x > \sin x$, for $x > 0$