Using Symmetry

(1) Even

$$f(-x) = f(x)$$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

NOTE: horizontal shift

$$\int_{c-a}^{c+a} f(x-c)dx = 2\int_{c}^{c+a} f(x-c)dx$$

(2) Odd

$$f(-x) = -f(x)$$

$$\int_{-a}^{a} f(x)dx = 0$$

NOTE: horizontal shift

$$\int_{c-a}^{c+a} f(x-c)dx = 0$$

(3)

$$\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$$

Proof:

$$\int_0^a f(a-x)dx$$

$$= -\int_a^0 f(u)du$$

$$= \int_0^a f(u)du$$

$$= \int_0^a f(x)dx$$

$$u = a - x$$

$$du = -dx$$

$$x = 0, u = a$$

$$x = a, u = 0$$

$$\text{odd} \times \text{odd} = \text{even}$$

$$\text{odd} \times \text{even} = \text{odd}$$

$$\text{even} \times \text{even} = \text{even}$$

$$\text{e.g. }(i)\int_{-1}^{1} \sin^3 x\,dx \quad = 0$$

$$(\text{odd function})^3 = \text{odd function}$$

$$(ii)\int_{0}^{1} x^2\sqrt{1-x}\,dx \quad = \int_{0}^{1} (1-x)^2\sqrt{x}\,dx$$

$$= \int_{0}^{1} \left(x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}} \right) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} \right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} - 0$$

$$= \frac{16}{105}$$

Improper Integrals

An **improper** integral is a definite integral where the integrand is undefined at some point in the interval or unbounded.

We must use limits to solve, if a solution exists.

e.g. (*i*)
$$\int_0^1 \frac{dx}{x} = \lim_{N \to 0} \int_N^1 \frac{dx}{x}$$
$$= \lim_{N \to 0} \left[\log x\right]_N^1$$
$$= \lim_{N \to 0} \left(-\log N\right)$$

Undefined

$\therefore$ integral is undefined

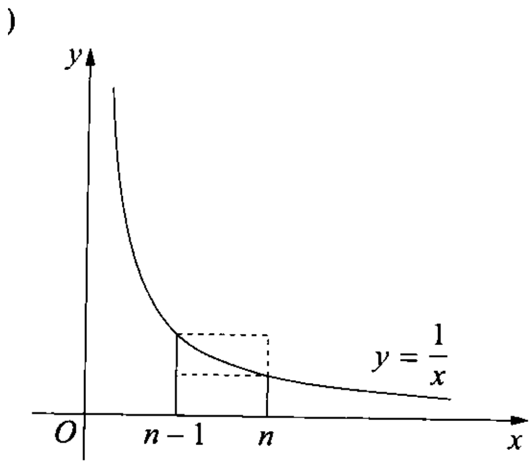
(*ii*)
$$\int_1^\infty xe^{-x^2}\,dx = \lim_{N \to \infty} \int_1^N xe^{-x^2}\,dx$$
$$= \lim_{N \to \infty} -\frac{1}{2}\left[e^{-x^2}\right]_1^N$$
$$= \lim_{N \to \infty} -\frac{1}{2}\left(e^{-N^2} - e^{-1}\right)$$
$$= \frac{1}{2e}$$

Area and Inequalities

If $g(x) \le f(x) \le h(x)$ for $a \le x \le b$

then $\int_a^b g(x)dx \le \int_a^b f(x)dx \le \int_a^b h(x)dx$

e.g. (2009 Question 8 b)



Let n be a positive integer greater than 1.

The area of the region under the curve $y = \frac{1}{x}$ from $x = n - 1$ to $x = n$ is between the area of two rectangles, as shown in the diagram.

Show that $e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}$

Area of inner rectangle < Area under the curve < Area of outer rectangle

$$(1)\left(\frac{1}{n}\right) < \int_{n-1}^{n} \frac{dx}{x} < (1)\left(\frac{1}{n-1}\right)$$

$$\frac{1}{n} < \left[\log x\right]_{n-1}^{n} < \frac{1}{n-1}$$

$$\frac{1}{n} < \log\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$$

$$1 < n\log\left(\frac{n}{n-1}\right) < \frac{n}{n-1}$$

$$1 < \log\left(\frac{n}{n-1}\right)^{n} < \frac{n}{n-1}$$

as $y = e^{x}$ is continually increasing

$$e < \left(\frac{n}{n-1}\right)^{n} < e^{\frac{n}{n-1}}$$

$$\frac{1}{e} > \left(\frac{n-1}{n}\right)^{n} > \frac{1}{e^{\frac{n}{n-1}}}$$

i.e. $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^{n} < e^{-1}$

Old Cambridge **Exercise 2I; 1, 2ac, 3, 6, 8, 9c, 10, 11ad, 12c, 13, 18**

Note: (2a – x) instead of (a – x)

Exercise 4I;1, 2defgh, 3a to h, 4, 6, 7b, 8, 12, 15

The 100 *(not 78)*