Integration By Parts

DIFF	INT	+ /–
u	dv	
$\frac{du}{dx}$	v	+
	$\int v\,dx$	–

$$\int u\,dv = uv - \int v\,du$$

u should be chosen so that differentiation makes it a simpler function.

dv should be chosen so that it can be integrated

Case 1: polynomial times integratable function

Differentiate the polynomial down to zero

e.g. $(i)\ \int x\cos x\,dx$

$$= x\sin x \;-\; -\cos x$$

$$= x\sin x + \cos x + c$$

DIFF	INT	+ /–
x	$\cos x$	
1	$\sin x$	+
0	$-\cos x$	–

$(ii)\int x^2 e^x\,dx$

$$= x^2e^x \;-2xe^x + 2e^x + c$$

DIFF	INT	+ /–
x^2	e^x	
$2x$	e^x	+
2	e^x	–
0	e^x	+

Case 2: polynomial times non-integratable function

Stop when product of the line is integratable

$$(i)\int \log x\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + c$$

DIFF	INT	+ /–
$\ln x$	1	
$\frac{1}{x}$	x	+
		–

$$(ii)\int \tan^{-1} x\,dx$$

$$= x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$$

$$= x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$$

DIFF	INT	+ /–
$\tan^{-1} x$	1	
$\frac{1}{1+x^2}$	x	+
		–

$$(iii)\int x^3 \log x dx$$

$$= \frac{1}{4}x^4 \log x - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4 \log x - \frac{1}{16}x^4 + c$$

DIFF	INT	+ /–
$\log x$	x^3	
$\frac{1}{x}$	$\frac{1}{4}x^4$	+
		–

OR

$$\int x^3 \log x dx$$

$$= x^4 \log x - x^4 - 3\int \left(x^3 \log x - x^3\right) dx$$

$$\therefore 4\int x^3 \log x dx = x^4 \log x - x^4 + 3\int x^3 dx$$

$$= x^4 \log x - x^4 + \frac{3}{4}x^4 + c$$

$$= x^4 \log x - \frac{1}{4}x^4 + c$$

$$\int x^3 \log x dx = \frac{1}{4}x^4 \log x - \frac{1}{16}x^4 + c$$

DIFF	INT	+ /–
x^3	$\log x$	
$3x^2$	$x \log x - x$	+
		–

Case 3: two integratable functions

Stop when product of the line is integratable or a multiple of another line

$$(i)\int e^{x}\cos x\,dx$$

$$= e^{x}\sin x + e^{x}\cos x - \int e^{x}\cos x\,dx$$

$$\therefore 2\int e^{x}\cos x\,dx = e^{x}\sin x + e^{x}\cos x + c$$

$$\int e^{x}\cos x\,dx = \frac{1}{2}e^{x}\sin x + \frac{1}{2}e^{x}\cos x + c$$

DIFF	INT	+ /–
e^{x}	$\cos x$	
e^{x}	$\sin x$	+
e^{x}	$-\cos x$	–
		+

Other examples

$$\int xe^{xdx}$$

$$= xe^{x} - e^{x} + c$$

DIFF	INT	+ /–
x	e^{x}	
1	e^{x}	+
0	e^{x}	–

$$\int \sin^{-1} x dx$$

$$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^{2}}} dx$$

$$= x\sin^{-1} x + \sqrt{1-x^{2}} + c$$

DIFF	INT	+ /–
$\sin^{-1} x$	1	
$\frac{1}{\sqrt{1-x^{2}}}$	x	+
		–

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx$$

$$= -\frac{\sqrt{1-x^2}}{x} - \int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x + c$$

DIFF	INT	+ /–
$\sqrt{1-x^2}$	$\frac{1}{x^2}$	
$-\frac{x}{\sqrt{1-x^2}}$	$-\frac{1}{x}$	+
		–

Using Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$

e.g. **2020 Extension 2 HSC Question 13d)**

(i) Show that for any integer n, $e^{in\theta} + e^{-in\theta} = 2\cos(n\theta)$

$$e^{in\theta} + e^{-in\theta} = \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta)$$

but $\cos(-n\theta) = \cos(n\theta)$ (even function)

$\sin(-n\theta) = -\sin(n\theta)$ (odd function)

$$\therefore e^{in\theta} + e^{-in\theta} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$

$$= \underline{2\cos(n\theta)}$$

(ii) By expanding $\left(e^{i\theta}+e^{-i\theta}\right)^4$, show that

$$\cos^4\theta=\frac{1}{8}(\cos(4\theta)+4\cos(2\theta)+3)$$

$$\left(e^{i\theta}+e^{-i\theta}\right)^4=e^{4i\theta}+4e^{3i\theta}e^{-i\theta}+6e^{2i\theta}e^{-2i\theta}+4e^{i\theta}e^{-3i\theta}+e^{-4i\theta}$$

$$=e^{4i\theta}+e^{-4i\theta}+4\left(e^{2i\theta}+e^{-2i\theta}\right)+6$$

$$\therefore(2\cos\theta)^4=2\cos(4\theta)+8\cos(2\theta)+6$$

$$16\cos^4\theta=2\cos(4\theta)+8\cos(2\theta)+6$$

$$\cos^4\theta=\frac{1}{8}(\cos(4\theta)+4\cos(2\theta)+3)$$

(iii) Hence, or otherwise, find $\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$

$$\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} \{\cos(4\theta) + 4\cos(2\theta) + 3\} \, d\theta$$

$$= \frac{1}{8}\left[\frac{1}{4}\sin(4\theta) + 2\sin(2\theta) + 3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{3\pi}{2}\right)$$

$$= \frac{3\pi}{16}$$

e.g.

$$\int e^x \cos x\,dx = Re\left(\int e^x(\cos x + i\sin x)dx\right)$$

$$= Re\left(\int e^x e^{ix}\,dx\right)$$

$$= Re\left(\int e^{(1+i)x}\,dx\right)$$

$$= Re\left(\frac{1}{1+i}e^{(1+i)x}\right) + c$$

$$= \frac{1}{2}Re\left((1-i)e^{(1+i)x}\right) + c$$

$$= \frac{1}{2}e^x\,Re((1-i)(\cos x + i\sin x)) + c$$

$$= \frac{1}{2}e^x(\cos x + \sin x) + c$$