Applications of Arithmetic & Geometric Series

e.g. (i) 2007 HSC Question 3b)

Heather decides to swim every day to improve her fitness level.

On the first day she swims 750 metres, and on each day after that she swims 100 metres more than the previous day.

That is she swims 850 metres on the second day, 950 metres on the third day and so on.

a) Write down a formula for the distance she swims on he *n*th day.

$$a = 750 \qquad T_n = a + (n-1)d$$

$$d = 100 \qquad T_n = 750 + (n-1)100$$

$$\underline{T_n = 650 + 100n}$$

b) How far does she swim on the 10th day?

$$T_n = 650 + 100n$$
$$T_{10} = 650 + 100(10)$$
$$T_{10} = 1650$$

$\therefore$ she swims 1650 metres on the 10th day

c) What is the total distance she swims in the first 10 days?

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$
$$S_{10} = \frac{10}{2}\{2(750) + 9(100)\}$$
$$S_{10} = 5(1500 + 900)$$
$$= 12000$$

OR

$$S_n = \frac{n}{2}(a + l)$$
$$S_{10} = \frac{10}{2}(750 + 1650)$$
$$= 12000$$

$\therefore$ she swims a total of 12000 metres in 10 days

d) After how many days does the total distance she has swum equal the width of the English Channel, a distance of 34 kilometres?

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$34000 = \frac{n}{2}\{1500 + (n-1)100\}$$

$$68000 = n(1400 + 100n)$$

$$68000 = 1400n + 100n^2$$

$$100n^2 + 1400n - 68000 = 0$$

$$n^2 + 14n - 680 = 0$$

$$(n+34)(n-20) = 0$$

$$\therefore n = -34 \quad \text{or} \quad n = 20$$

$\therefore$ it takes 20 days to swim 34 kilometres

(ii) A company's sales are declining by 6% every year, with 50000 items sold in 2001.

During which year will sales first fall below 20000?

$$a = 50000$$
$$r = 0.94$$
$$T_n < 20000$$

$$T_n = ar^{n-1}$$
$$20000 > 50000(0.94)^{n-1}$$
$$0.4 > (0.94)^{n-1}$$
$$(0.94)^{n-1} < 0.4$$
$$\log(0.94)^{n-1} < \log 0.4$$
$$(n-1)\log(0.94) < \log 0.4$$
$$(n-1) > \frac{\log 0.4}{\log 0.94}$$
$$(n-1) > 14.80864248$$
$$n > 15.80864248$$

$\therefore$ during the 16th year (i.e. 2016) sales will fall below 20000

(iii) 2005 HSC Question 7a)

Anne and Kay are employed by an accounting firm.

Anne accepts employment with an initial annual salary of \$50000. In each of the following years her annual salary is increased by \$2500.

Kay accepts employment with an initial annual salary of \$50000. in each of the following years her annual salary is increased by 4%

arithmetic

$a = 50000$ $\qquad d = 2500$

geometric

$a = 50000$ $\qquad r = 1.04$

a) What is Anne's salary in her thirteenth year?

$$T_n = a + (n-1)d$$

$$T_{13} = 50000 + 12(2500)$$

$$T_{13} = 80000$$

$\therefore$ Anne earns \$80000 in her 13th year

b) What is Kay's annual salary in her thirteenth year?

$$T_n = ar^{n-1}$$
$$T_{13} = 50000(1.04)^{12}$$
$$T_{13} = 80051.61093\ldots$$

$\therefore$ Kay earns \$80051.61 in her thirteenth year

c) By what amount does the total amount paid to Kay in her first twenty years exceed that paid to Anne in her first twenty years?

Anne
$$S_n = \frac{n}{2}\{2a+(n-1)d\}$$
$$S_{20} = \frac{20}{2}\{2(50000)+19(2500)\}$$
$$S_{10} = 10(100000+47500)$$
$$= 1475000$$

Kay
$$S_n = \frac{a(r^n-1)}{n-1}$$
$$S_{20} = \frac{50000(1.04^{20}-1)}{0.04}$$
$$= 1488903.93$$

$\therefore$ Kay is paid \$13903.93 more than Anne

(iv) Explain why the geometric series

$$2 + \frac{2}{\sqrt{2} - 1} + \frac{2}{(\sqrt{2} - 1)^2} + \ldots \text{ does } NOT \text{ have a limiting sum.}$$

Limiting sums only occur when $-1 < r < 1$

$$r = \frac{2}{\sqrt{2}-1} \times \frac{1}{2}$$

$$= \frac{1}{\sqrt{2}-1}$$

$$\therefore r = 2.414\ldots > 1$$

as $r > 1$, no limiting sum exists

(v) 2004 HSC Question 9a)

Consider the series $1 - \tan^2\theta + \tan^4\theta - \ldots$

a) When the limiting sum exists, find its value in simplest form.

$$a = 1$$

$$r = -\tan^2\theta$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{1}{1+\tan^2\theta}$$

$$= \frac{1}{\sec^2\theta}$$

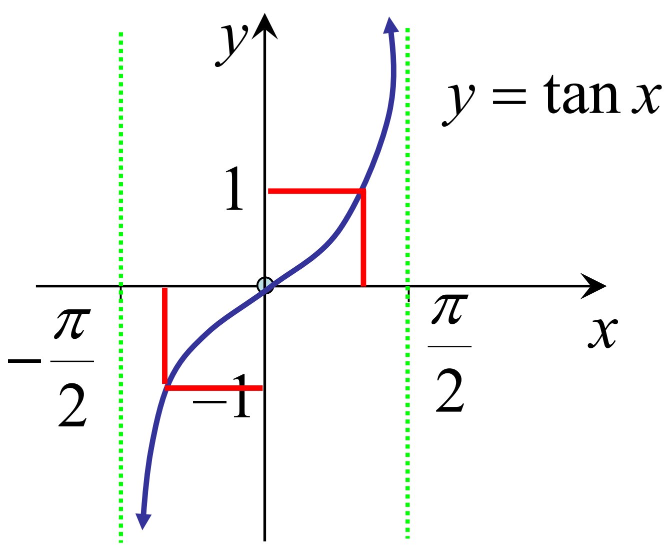
$$= \cos^2\theta$$

b) For what values of θ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ does the limiting sum of the series exist?

Limiting sums only occur when $-1 < r < 1$

$$-1 < -\tan^2\theta < 1$$
$$1 > \tan^2\theta > -1$$
$$-1 < \tan^2\theta < 1$$
$$-1 < \tan\theta < 1$$

Now $\tan\theta = \pm 1$, when $\theta = \pm\frac{\pi}{4}$



$$\therefore -\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

Exercise 14A; 3, 4, 6, 8, 10, 13, 14, 15

Exercise 14B; 4, 6, 7, 9, 10, 12, 13