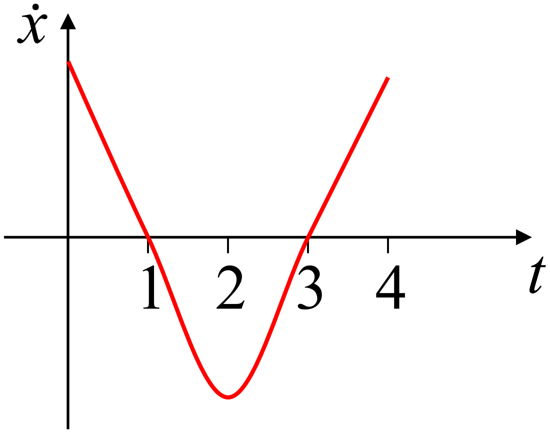
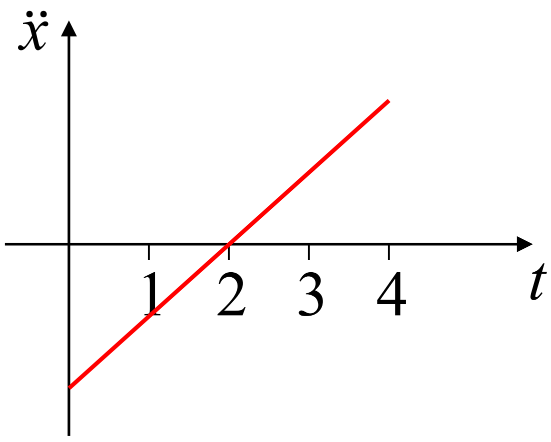
Integrating Functions of Time



$$\text{change in displacement} = \int_0^4 \dot{x}\,dt$$

$$\text{change in distance} = \int_0^1 \dot{x}\,dt - \int_1^3 \dot{x}\,dt + \int_3^4 \dot{x}\,dt$$



$$\text{change in velocity} = \int_0^4 \ddot{x}\,dt$$

$$\text{change in speed} = -\int_0^2 \ddot{x}\,dt + \int_2^4 \ddot{x}\,dt$$

Derivative Graphs

Function *displacement*	**1st derivative** *velocity*	**2nd derivative** *acceleration*
stationary point	x intercept	
inflection point	stationary point	x intercept
increasing	positive	
decreasing	negative	
concave up	increasing	positive
concave down	decreasing	negative

graph type	integrate	differentiate
horizontal line	oblique line	x axis
oblique line	parabola	horizontal line
parabola	cubic *inflects at turning pt*	oblique line

Remember:

- integration = area
- on a velocity graph, total area = distance
  total integral = displacement
- on an acceleration graph, total area = speed
  total integral = velocity

a) **2003 HSC Question 7b)**

The velocity of a particle is given by $v = 2 - 4\cos t$ for $0 \leq t \leq 2\pi$, where v is measured in metres per second and t is measured in seconds

(i) At what times during this period is the particle at rest?

$$v = 0$$
$$2 - 4\cos t = 0$$
$$\cos t = \frac{1}{2}$$

Q1, 4

$$\cos\alpha = \frac{1}{2}$$
$$\alpha = \frac{\pi}{3}$$

$$t = \alpha, 2\pi - \alpha$$
$$t = \frac{\pi}{3}, \frac{5\pi}{3}$$

$\therefore$ particle is at rest after $\frac{\pi}{3}$ seconds and again after $\frac{5\pi}{3}$ seconds

(ii) What is the maximum velocity of the particle during this period?

$$-4 \leq -4\cos t \leq 4$$
$$-2 \leq 2 - 4\cos t \leq 6$$

$\therefore$ maximum velocity is 6 m/s

(iii) Sketch the graph of v as a function of t for $0 \leq t \leq 2\pi$

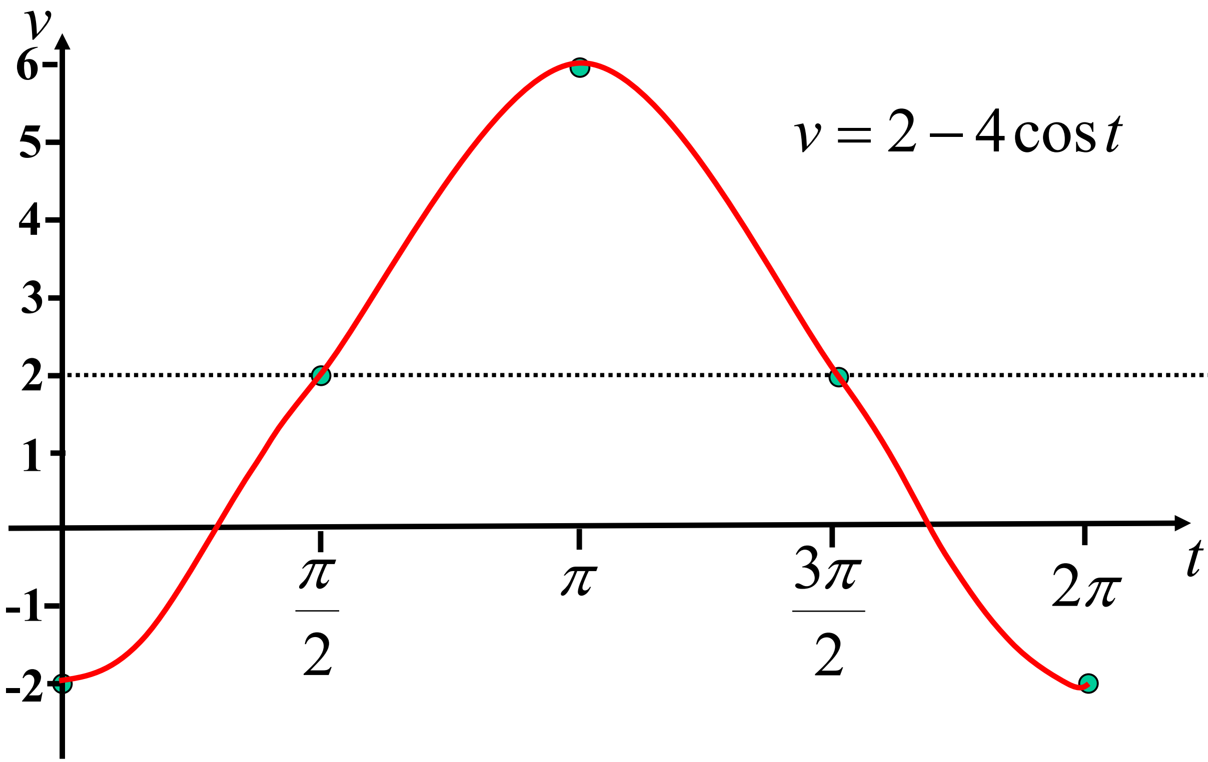
amplitude = 4 units

shift = $\uparrow$ 2 units

flip upside down

$$\text{period} = \frac{2\pi}{1} = 2\pi$$

$$\text{divisions} = \frac{2\pi}{4} = \frac{\pi}{2}$$



(iv) Calculate the total distance travelled by the particle between $t = 0$ and $t = \pi$

$$\text{distance} = -\int_0^{\frac{\pi}{3}} (2 - 4\cos t)\,dt + \int_{\frac{\pi}{3}}^{\pi} (2 - 4\cos t)\,dt$$

$$= \left[2t - 4\sin t\right]_{\frac{\pi}{3}}^{0} + \left[2t - 4\sin t\right]_{\frac{\pi}{3}}^{\pi}$$

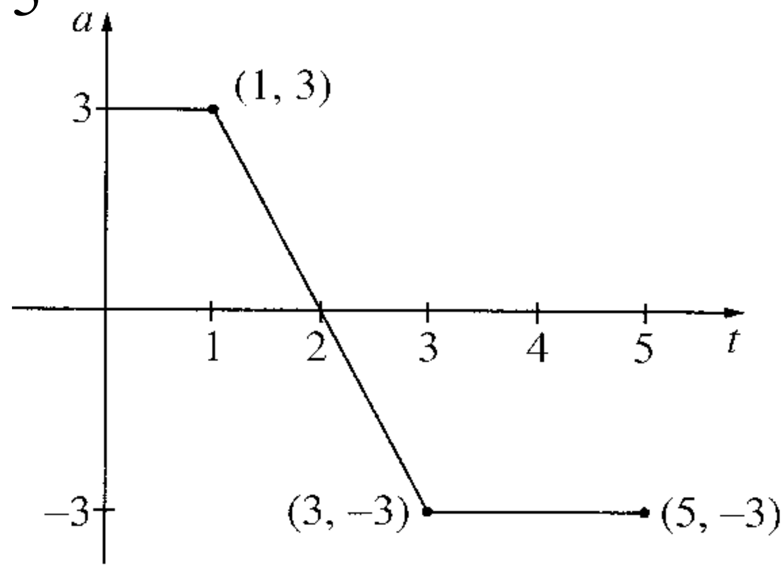
$$= (0 - 0) + (2\pi - 4\sin\pi) - 2\left(\frac{2\pi}{3} - 4\sin\frac{\pi}{3}\right)$$

$$= 2\pi - 2\left(\frac{2\pi}{3} - \frac{4\sqrt{3}}{2}\right)$$

$$= 4\sqrt{3} + \frac{2\pi}{3} \text{ metres}$$

b) **2004 HSC Question 9b)**

A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration, a, of the particle as a function of time t for $0 \leq t \leq 5$



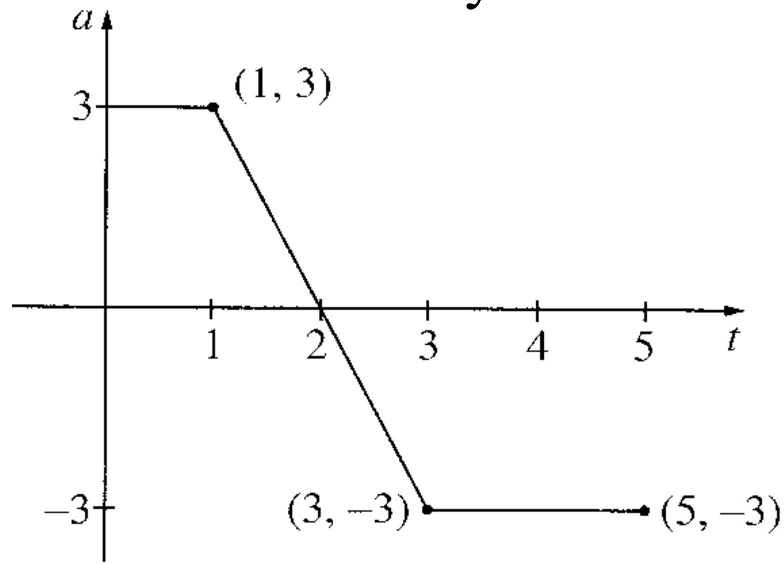
(i) Write down the time at which the velocity of the particle is a maximum

$v = \int a\,dt$ ***OR*** v is a maximum when $\frac{dv}{dt} = 0$

$\int a\,dt$ is a maximum when $t = 2$

$\therefore$ velocity is a maximum when $t = 2$ seconds

(ii) At what time during the interval $0 \leq t \leq 5$ is the particle furthest from the origin? Give reasons for your answer.



Question is asking, "when is displacement a maximum?"

$$x \text{ is a maximum when } \frac{dx}{dt} = 0$$

$$\text{But } v = \int a\,dt$$

$$\therefore \text{ We must solve } \int a\,dt = 0$$

i.e. when is area above the axis = area below

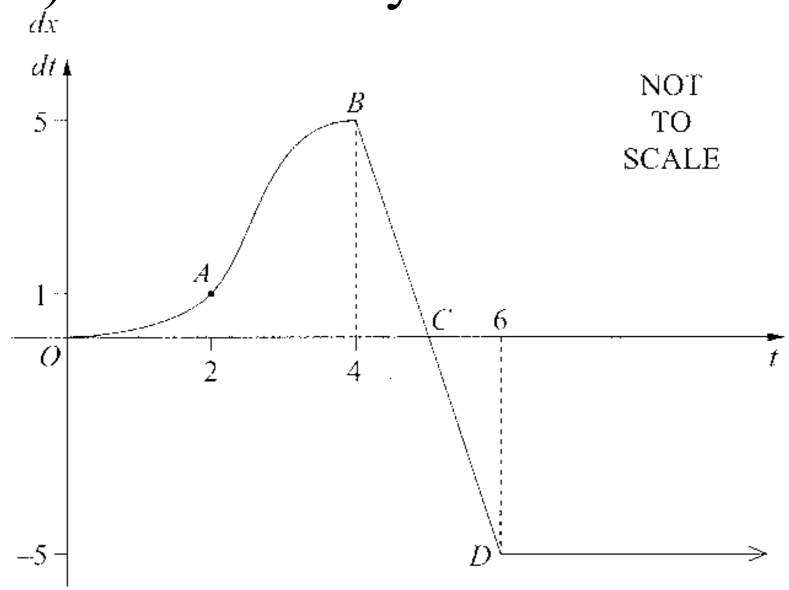
By symmetry this would be at $t = 4$

$\therefore$ particle is furthest from the origin at $t = 4$ seconds

c) **2007 HSC Question 10a)**

An object is moving on the x-axis. The graph shows the velocity, $\frac{dx}{dt}$, of the object, as a function of t.

The coordinates of the points shown on the graph are $A(2,1)$, $B(4,5)$, $C(5,0)$ and $D(6,-5)$. The velocity is constant for $t \geq 6$



(i) Using the Trapezoidal Rule, estimate the distance travelled between $t = 0$ and $t = 4$

	$\boxed{1}$	$\boxed{2}$	$\boxed{1}$
t	0	2	4
v	0	1	5

$$\text{distance} \approx \frac{h}{2}\{y_0 + 2y_{\text{others}} + y_n\}$$

$$= \frac{2}{2}\{0 + 2(1) + 5)$$

$$= 7 \text{ metres}$$

(ii) The object is initially at the origin. During which time(s) is the displacement decreasing?

$$x \text{ is decreasing when } \frac{dx}{dt} < 0$$

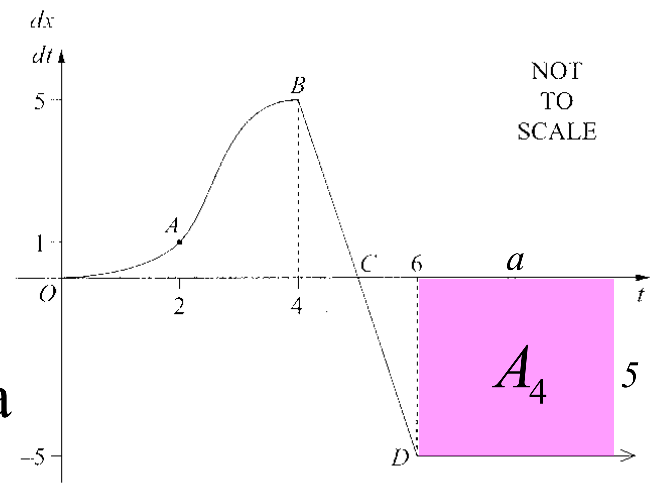
$\therefore$ displacement is decreasing when $t > 5$ seconds

(iii) Estimate the time at which the object returns to the origin. Justify your answer.

Question is asking, "when is displacement = 0?"

$$\text{But } x = \int v\,dt$$

$$\therefore \text{We must solve } \int v\,dt = 0$$



i.e. when is area above the axis = area below

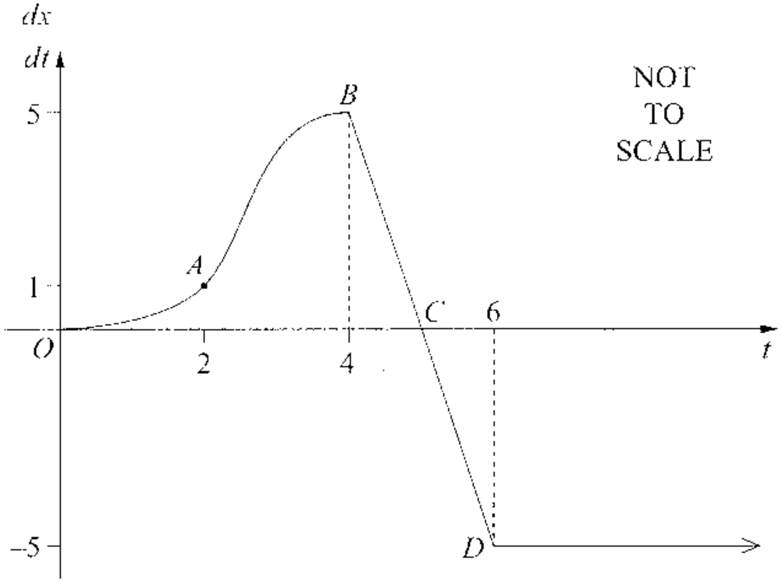
By symmetry, area from $t = 4$ to 5 equals area from $t = 5$ to 6

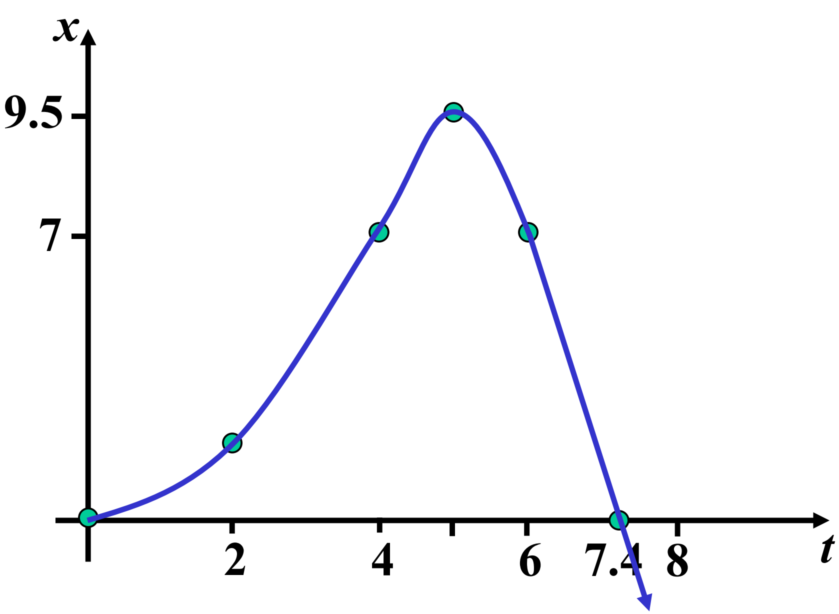
In part (i) we estimated area from $t = 0$ to 4 to be 7,

$\therefore A_4 = 7$ $\qquad a = 1.4$

$5a = 7$ $\qquad \therefore$ particle returns to the origin when $t = 7.4$ seconds

(iv) Sketch the displacement, x, as a function of time.





object is initially at the origin

when $t = 4$, $x = 7$

by symmetry of areas $t = 6$, $x = 7$

Area of triangle = 2.5

$\therefore$ when $t = 5$, $x = 9.5$

returns to $x = 0$ when $t = 7.4$

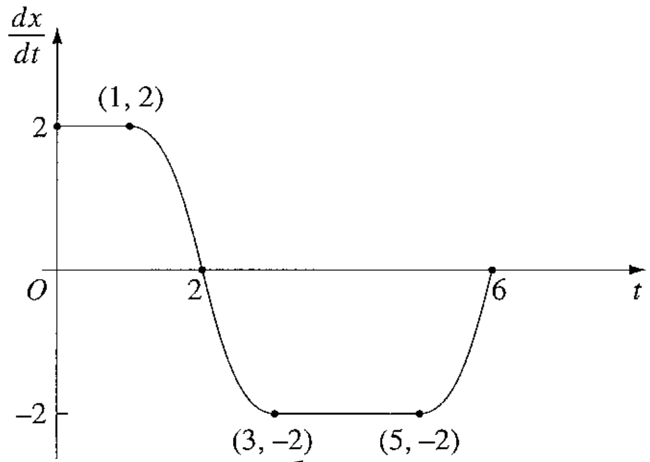
v is steeper between $t = 2$ and 4 than between $t = 0$ and 2

$\therefore$ particle covers more distance between $t = 2$ and 4

when $t > 6$, v is constant

$\therefore$ when $t > 6$, x is a straight line

d) **2005 HSC Question 7b)**



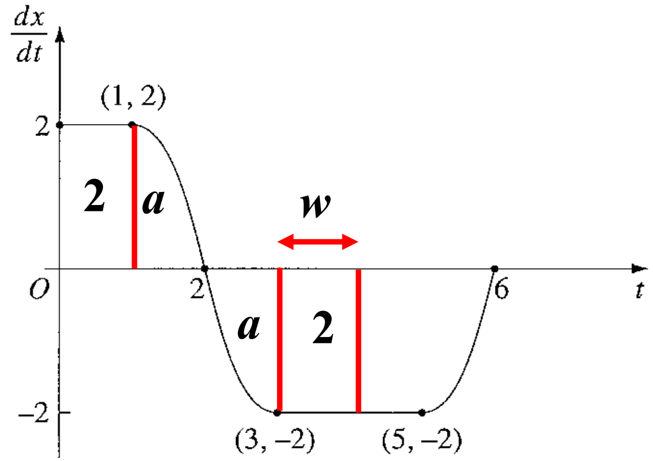
The graph shows the velocity, $\frac{dx}{dt}$, of a particle as a function of time. Initially the particle is at the origin.

(i) At what time is the displacement, x, from the origin a maximum?

Displacement is a maximum when area is most positive, also when velocity is zero

i.e. when $t = 2$

(ii) At what time does the particle return to the origin? Justify your answer



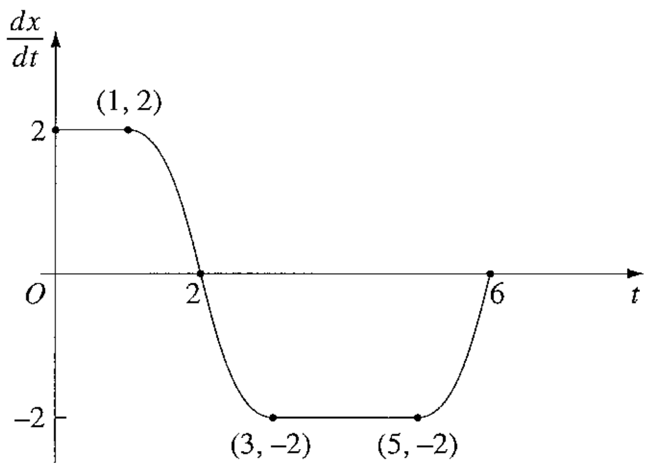
Question is asking, "when is displacement = 0?"

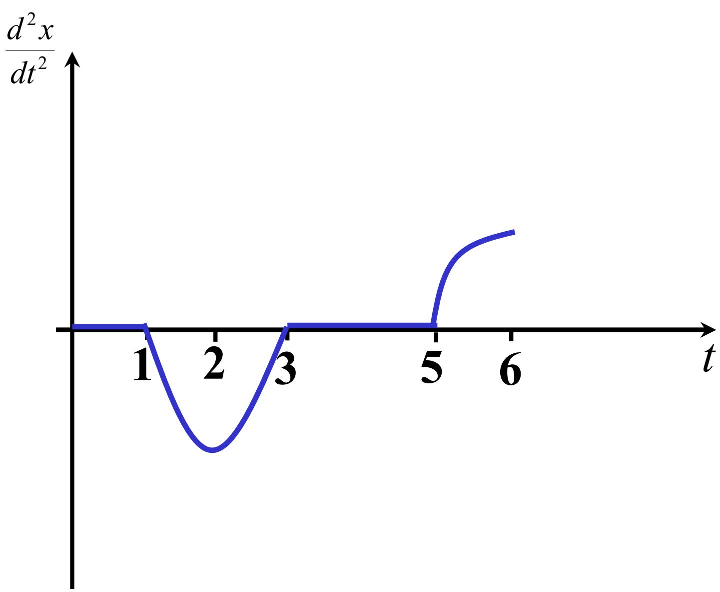
i.e. when is area above the axis = area below?

$$2w = 2$$
$$w = 1$$

Returns to the origin after 4 seconds

(iii) Draw a sketch of the acceleration, $\frac{d^2x}{dt^2}$, as afunction of time for $0 \leq t \leq 6$





differentiate a horizontal line you get the $x$axis

from 1 to 3 we have a cubic, inflects at 2, and is decreasing

differentiate, you get a parabola, stationary at 2, it is below the x axis

from 5 to 6 is a cubic, inflects at 6 and is increasing *(using symmetry)*

differentiate, you get a parabola stationary at 6, it is above the x axis

Exercise 9C; 3, 4, 6, 10, 12, 13, 15, 17a, 18