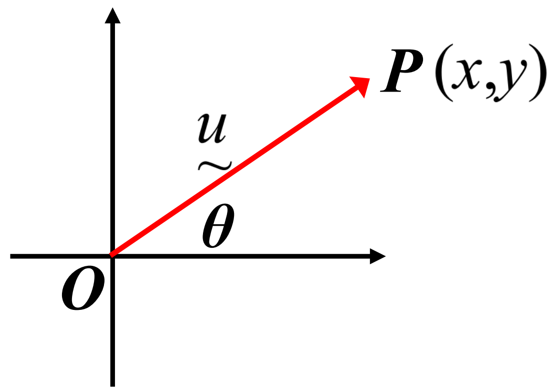
Position Vectors



$$\underset{\sim}{u} = \overrightarrow{OP} = (x,y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

position vector	***ordered pair***	***column vector***	***component form***

$$|\underset{\sim}{u}| = \sqrt{x^2 + y^2}$$

$$\text{direction} = \theta$$

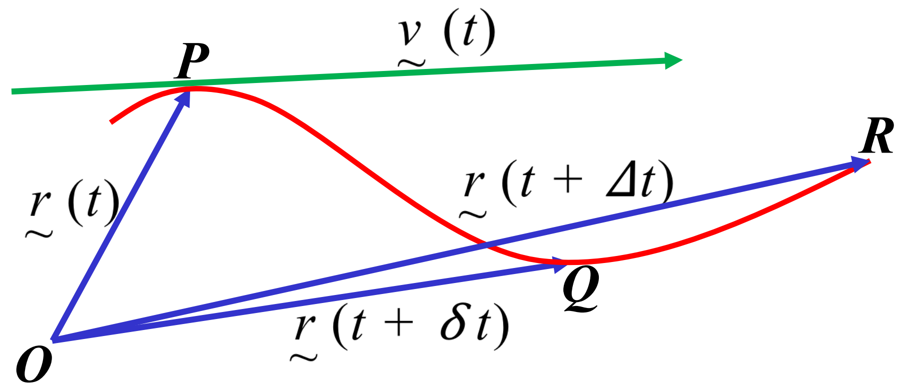
$$= \tan^{-1}\frac{y}{x}$$

Space Curves

So far all of our motion has been motion in a straight line

Consider a **position vector**, $\underset{\sim}{r}(t)$, as a function of a scalar, t

As t varies $\underset{\sim}{r}$ describes a **space curve**



The **velocity vector,** $\underset{\sim}{v}(t)$, is tangential to the space curve at *P*.

To find velocity and acceleration, and their components, for objects moving in curvilinear paths, the use of vector analysis is needed.

displacement $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$
velocity $\underset{\sim}{v}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$ $\lvert\underset{\sim}{v}(t)\rvert = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$ direction $= \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right)$
acceleration $\underset{\sim}{a}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$ $\lvert\underset{\sim}{a}(t)\rvert = \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2}$

e.g. (i) A particle moves so that its velocity at time t is given by

$$\underset{\sim}{v}(t) = -2\sin 2t\underset{\sim}{i} + 4\cos 2t\underset{\sim}{j} \quad \text{for } 0 \leq t \leq \frac{\pi}{2}$$

Given that $\underset{\sim}{r}(0) = \underset{\sim}{i}$, find the position vector of the particle at any time t.

$$\frac{dx}{dt} = -2\sin 2t$$

$$\int_1^x dx = -2\int_0^t \sin 2t\,dt$$

$$\left[x\right]_1^x = \left[\cos 2t\right]_0^t$$

$$x - 1 = \cos 2t - 1$$

$$x = \cos 2t$$

$$\frac{dy}{dt} = 4\cos 2t$$

$$\int_0^y dy = 4\int_0^t \cos 2t$$

$$y = 2\left[\sin 2t\right]_0^t$$

$$y = 2\sin 2t$$

$$\underset{\sim}{r}(t) = \cos 2t\underset{\sim}{i} + 2\sin 2t\underset{\sim}{j}$$

(ii) The position vectors, at time t, of particles A and B are given by;

$$\underset{\sim}{r}_A(t) = (t^3 - 9t + 8)\underset{\sim}{i} + t^2\underset{\sim}{j}$$

$$\underset{\sim}{r}_B(t) = (2 - t^2)\underset{\sim}{i} + (3t - 2)\underset{\sim}{j}$$

Prove that A and B collide while travelling at the same speed, but at right angles to each other.

Particles will collide when they are at the same position at the same time

$$t^3 - 9t + 8 = 2 - t^2$$

$$t^3 + t^2 - 9t + 6 = 0$$

$$t^2 = 3t - 2$$

$$t^2 - 3t + 2 = 0$$

$$(t - 2)(t - 1) = 0$$

$$t = 1 \text{ or } t = 2$$

when $t = 1$, $t^3 + t^2 - 9t + 6 = -1$

when $t = 2$, $t^3 + t^2 - 9t + 6 = 0$

$\therefore$ the two particles collide after 2 seconds

$$\underset{\sim}{r}_A = (t^3 - 9t + 8)\underset{\sim}{i} + t^2\underset{\sim}{j} \qquad \underset{\sim}{r}_B = (2 - t^2)\underset{\sim}{i} + (3t - 2)\underset{\sim}{j}$$

$$\dot{\underset{\sim}{r}}_A = (3t^2 - 9)\underset{\sim}{i} + 2t\underset{\sim}{j} \qquad \dot{\underset{\sim}{r}}_B = -2t\underset{\sim}{i} + 3\underset{\sim}{j}$$

when $t = 2$; $\dot{\underset{\sim}{r}}_A = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ $\qquad \dot{\underset{\sim}{r}}_B = -4\underset{\sim}{i} + 3\underset{\sim}{j}$

$$\left|\dot{\underset{\sim}{r}}_A\right| = \sqrt{3^2 + 4^2} = 5 \qquad \left|\dot{\underset{\sim}{r}}_B\right| = \sqrt{(-4)^2 + 3^2} = 5$$

$\therefore$ when the particles collide they are both travelling at 5 m/s

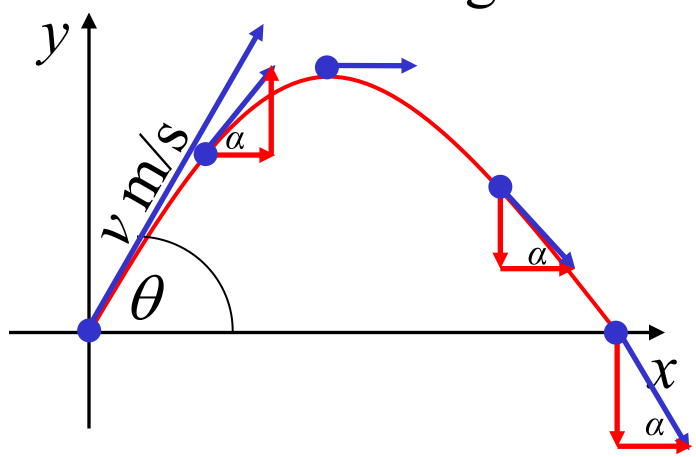
$$\dot{\underset{\sim}{r}}_A \cdot \dot{\underset{\sim}{r}}_B = \left(3\underset{\sim}{i} + 4\underset{\sim}{j}\right)\cdot\left(-4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$
$$= (3)(-4) + (4)(3)$$
$$= 0$$

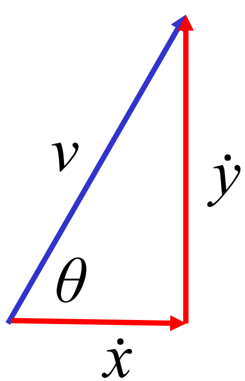
$\therefore$ the particles collide at right angles to each other

Projectile Motion

In Extension 1 we consider the two-dimensional motion of a particle projected into the air, subject to gravity only (i.e. we assume that air resistance is negligible), $\ddot{x}=0 \qquad \ddot{y}=-g\underset{\sim}{j}$

<u>Initial conditions</u> when $t=0$; particle is projected with a velocity, v ms^{-1}, at an angle of θ to the horizontal





$$\frac{\dot{x}}{v}=\cos\theta \qquad \frac{\dot{y}}{v}=\sin\theta$$

$$\dot{x}=v\cos\theta \qquad \dot{y}=v\sin\theta$$

$$x=0 \qquad y=0$$

Note:
maximum range

$$\theta=45^{\circ}$$

$$\frac{d\dot{x}}{dt} = 0 \qquad\qquad \frac{d\dot{y}}{dt} = -10$$

$$\int_{v\cos\theta}^{\dot{x}} d\dot{x} = 0 \qquad\qquad \int_{v\sin\theta}^{\dot{y}} d\dot{y} = -10\int_{0}^{t} dt$$

$$\left[\dot{x}\right]_{v\cos\theta}^{\dot{x}} = 0 \qquad\qquad \left[\dot{y}\right]_{v\sin\theta}^{\dot{y}} = -10t$$

$$\dot{x} - v\cos\theta = 0 \qquad\qquad \dot{y} - v\sin\theta = -10t$$

$$\dot{x} = v\cos\theta \qquad\qquad \dot{y} = v\sin\theta - 10t$$

$$\underset{\sim}{v} = |\underset{\sim}{v}|\cos\theta\,\underset{\sim}{i} + (|\underset{\sim}{v}|\sin\theta - 10t)\underset{\sim}{j}$$

$$\int_{0}^{x} dx = \int_{0}^{t} v\cos\theta\, dt \qquad\qquad \int_{0}^{y} dy = \int_{0}^{t} (v\sin\theta - 10t)dt$$

$$x = \left[vt\cos\theta\right]_{0}^{t} \qquad\qquad y = \left[vt\sin\theta - 5t^2\right]_{0}^{t}$$

$$x = vt\cos\theta \qquad\qquad y = vt\sin\theta - 5t^2$$

$$\underset{\sim}{r} = |\underset{\sim}{v}|t\cos\theta\,\underset{\sim}{i} + \left(|\underset{\sim}{v}|t\sin\theta - 5t^2\right)\underset{\sim}{j}$$

Common Questions

(1) When does the particle hit the ground?

Particle hits the ground when $y = 0$

(2) What is the range of the particle?

(i) find when $y = 0$

(ii) substitute into x

(3) What is the greatest height of the particle?

(i) find when $\dot{y} = 0$

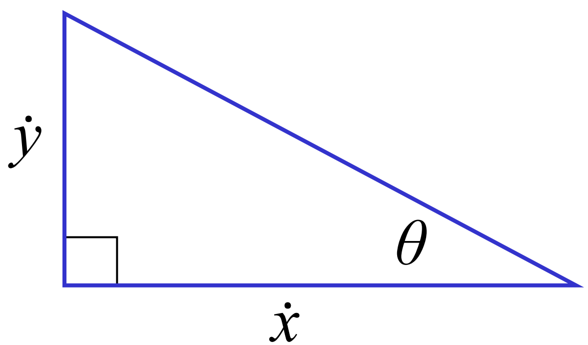
(ii) substitute into y

(4) What angle does the particle make with the ground?

(i) find when $y = 0$

(ii) substitute into $\dot{y}$

(iii) $\tan\theta = \frac{\dot{y}}{\dot{x}}$



Summary

A particle undergoing projectile motion obeys

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g$$

with initial conditions

$$\dot{x} = v\cos\theta \qquad \text{and} \qquad \dot{y} = v\sin\theta$$

e.g. A ball is thrown with an initial velocity of 25 m/s at an angle of $\theta = \tan^{-1}\frac{3}{4}$ to the ground. Determine;.

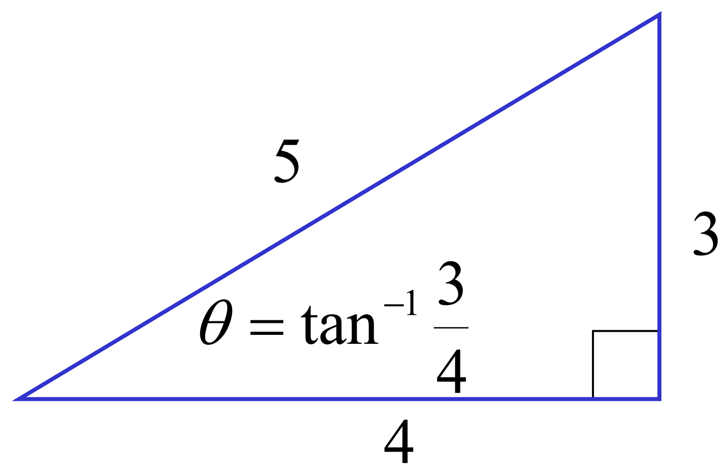
a) greatest height obtained

Initial conditions

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta$$

$$\dot{x} = 25\left(\frac{4}{5}\right) \qquad \dot{y} = 25\left(\frac{3}{5}\right)$$

$$= 20\text{m/s} \qquad = 15\text{m/s}$$



$$\ddot{x} = 0$$

$$\frac{d\dot{x}}{dt} = 0$$

$$\int_{20}^{\dot{x}} d\dot{x} = 0$$

$$\left[\dot{x}\right]_{20}^{\dot{x}} = 0$$

$$\dot{x} - 20 = 0$$

$$\dot{x} = 20$$

$$\frac{dx}{dt} = 20$$

$$\int_{0}^{x} dx = \int_{0}^{t} 20dt$$

$$\left[x\right]_{0}^{x} = \left[20t\right]_{0}^{t}$$

$$x = 20t$$

$$\ddot{y} = -10$$

$$\frac{d\dot{y}}{dt} = -10$$

$$\int_{15}^{\dot{y}} d\dot{y} = -\int_{0}^{t} 10dt$$

$$\left[\dot{y}\right]_{15}^{\dot{y}} = \left[10t\right]_{t}^{0}$$

$$\dot{y} - 15 = 0 - 10t$$

$$\dot{y} = 15 - 10t$$

$$\frac{dy}{dt} = 15 - 10t$$

$$\int_{0}^{y} dy = \int_{0}^{t} 15 - 10tdt$$

$$\left[y\right]_{0}^{y} = \left[15t - 5t^{2}\right]_{0}^{t}$$

$$y = 15t - 5t^{2}$$

greatest height occurs when $\dot{y} = 0$

$$-10t + 15 = 0$$

$$t = \frac{3}{2}$$

when $t = \frac{3}{2}$, $y = -5\left(\frac{3}{2}\right)^2 + 15\left(\frac{3}{2}\right)$

$$= \frac{45}{4}$$

$\therefore$ greatest height is $11\frac{1}{4}$ m above the ground

b) range

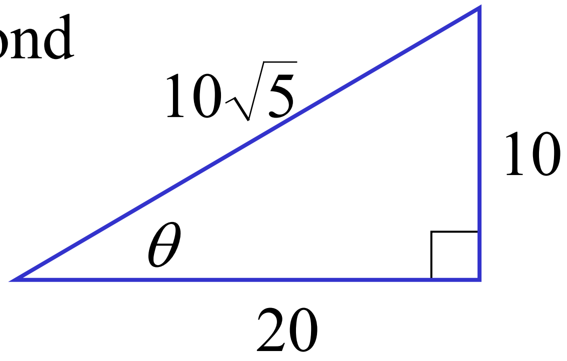
time of flight is 3 seconds

when $t = 3, x = 20(3)$

$$= 60$$

$\therefore$ range is 60m

c) velocity and direction of the ball after $\frac{1}{2}$ second



when $t = \frac{1}{2}, \dot{x} = 20$

$$\dot{y} = -10\left(\frac{1}{2}\right) + 15$$
$$= 10$$

$$\tan\theta = \frac{1}{2}$$

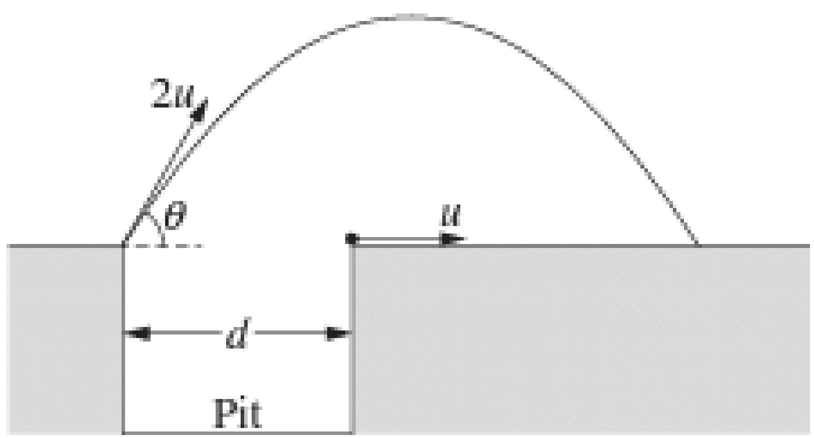
$$\theta = 26^\circ 34'$$

$\therefore$ after $\frac{1}{2}$ second, velocity $= 10\sqrt{5}$m/s and it is traveling at an angle of $26^\circ 34'$ to the horizontal

(ii) **2022 Extension 1 HSC Question 14 b)**

A video game designer wants to include an obstacle in the game they are developing. The player will reach one side of a pit and must shoot a projectile to hit a target on the other side of the pit in order to be able to cross. However, the instant the player shoots, the target begins to move away from the player at a constant speed that is half the initial speed of the projectile shot by the player, as hown in the diagram below.

The initial distance between the player and the target is d, the initial speed of the projectile is $2u$ and it is launched at an angle of θ to the horizontal. The acceleration due to gravity is g. The launch nagle is the ONLY parameter that the player can change.



Taking the position of the player when the projectile is launched as the origin, the positions of the projectile and target at time t after the projectile is launched are as follows.

$$\underset{r_p}{\rightarrow} = \begin{pmatrix} 2ut\cos\theta \\ 2ut\sin\theta - \frac{g}{2}t^2 \end{pmatrix} \qquad \text{Projectile}$$

$$\underset{r_T}{\rightarrow} = \begin{pmatrix} d + ut \\ 0 \end{pmatrix} \qquad \text{Target}$$

Show that, for the player to have a chance of hitting the target, d must be less than 37% of the maximum possible range of the projectile

projectile hits the ground when $\underset{r_p}{\rightarrow} = \begin{pmatrix} 2ut\cos\theta \\ 0 \end{pmatrix}$

i.e. $2ut\sin\theta - \frac{1}{2}gt^2 = 0$

$t\left(2u\sin\theta - \frac{1}{2}gt\right) = 0$

$t = 0$ or $t = \frac{4u\sin\theta}{g}$ $\quad\therefore$ projectile hits the ground when $t = \frac{4u\sin\theta}{g}$

$$\text{when } t = \frac{4u\sin\theta}{g}, \quad x_p = 2u \times \frac{4u\sin\theta}{g} \times \cos\theta \qquad x_T = d + u \times \frac{4u\sin\theta}{g}$$

$$= \frac{8u^2\sin\theta\cos\theta}{g} \qquad = d + \frac{4u^2\sin\theta}{g}$$

$$= \frac{4u^2\sin 2\theta}{g} \quad \Rightarrow \text{maximum range is } \frac{4u^2}{g}$$

In order to hit the target $x_T \leq x_p$

$$\text{i.e. } d + \frac{4u^2\sin\theta}{g} \leq \frac{4u^2\sin 2\theta}{g}$$

$$d \leq \frac{4u^2}{g}(\sin 2\theta - \sin\theta)$$

d will be maximised when $\sin 2\theta - \sin\theta$ is maximised

$$f(\theta) = \sin 2\theta - \sin\theta$$

$$f'(\theta) = 2\cos 2\theta - \cos\theta$$

$$= 2(2\cos^2\theta - 1) - \cos\theta$$

$$= 4\cos^2\theta - \cos\theta - 2$$

stationary points occur when $f'(\theta) = 0$

i.e. $4\cos^2\theta - \cos\theta - 2 = 0$

$$\cos\theta = \frac{1 \pm \sqrt{33}}{8}$$

$$\cos\theta = \frac{1 + \sqrt{33}}{8} \qquad (\theta \text{ is acute})$$

$$= 0.8431....$$

$$\sin\theta = \sqrt{1 - 0.8431^2}$$
$$= 0.5378...$$

$$d \le \frac{4u^2}{g}[2(0.5378)(0.8431) - 0.5378)]$$

$$= \frac{4u^2}{g}(0.3690)$$

$\therefore$ d must be 37% of the maximum range.

Exercise 10A; 1, 3, 5, 7, 9, 10, 11, 13, 14, 15, 16, 17