

Velocity & Acceleration in Terms of x

If $v = f(x)$;

$$\frac{d^2x}{dt^2} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

Proof:

$$\frac{d^2x}{dt^2} = \frac{dv}{dt}$$

$$= \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{dv}{dx} \cdot v$$

$$= \frac{dv}{dx} \cdot \frac{d}{dv} \left(\frac{1}{2} v^2 \right)$$

$$= \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

*Useful when you
want to link v and t*

*Useful when you
want to link v and x*

e.g. (i) A particle moves in a straight line so that $\ddot{x} = 3 - 2x$

Find its velocity in terms of x given that $v = 2$ when $x = 1$.

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 3 - 2x$$

$$\frac{1}{2} v^2 = 3x - x^2 + c$$

when $x = 1, v = 2$

$$\text{i.e. } \frac{1}{2} (2)^2 = 3(1) - 1^2 + c$$

$$c = 0$$

$$\therefore v^2 = 6x - 2x^2$$

$$\underline{v = \pm \sqrt{6x - 2x^2}}$$

OR

$$v \frac{dv}{dx} = 3 - 2x$$

$$\int_2^v v dv = \int_1^x (3 - 2x) dx$$

$$\left[\frac{1}{2} v^2 \right]_2^v = \left[3x - x^2 \right]_1^x$$

$$\frac{1}{2} v^2 - 2 = 3x - x^2 - 2$$

$$\therefore v^2 = 6x - 2x^2$$

NOTE: $v^2 \geq 0$

$$6x - 2x^2 \geq 0$$

$$2x(3 - x) \geq 0$$

$$\therefore 0 \leq x \leq 3$$

Particle moves between $x = 0$
and $x = 3$ and nowhere else.

(ii) A particle's acceleration is given by $\ddot{x} = 3x^2$. Initially, the particle is 1 unit to the right of O , and is traveling with a velocity of $\sqrt{2}\text{m/s}$ in the negative direction. Find x in terms of t .

$$v \frac{dv}{dx} = 3x^2$$

$$\frac{dx}{dt} = -\sqrt{2x^3}$$

$$= -\sqrt{2}x^{\frac{3}{2}}$$

Choose -ve to satisfy the initial conditions

$$\int_{-\sqrt{2}}^v v dv = \int_1^x 3x^2 dx$$

$$\int_0^t dt = -\frac{1}{\sqrt{2}} \int_1^x x^{-\frac{3}{2}} dx$$

$$t + \sqrt{2} = \sqrt{\frac{2}{x}}$$

$$\left[\frac{1}{2} v^2 \right]_{-\sqrt{2}}^v = \left[x^3 \right]_1^x$$

$$t = -\frac{1}{\sqrt{2}} \left[-2x^{-\frac{1}{2}} \right]_1^x$$

$$\frac{2}{x} = (t + \sqrt{2})^2$$

$$\frac{1}{2} v^2 - 1 = x^3 - 1$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{x}} - 1 \right)$$

$$x = \frac{2}{(t + \sqrt{2})^2}$$

$$v^2 = 2x^3$$

$$v = \pm \sqrt{2x^3}$$

(iii) 2004 Extension 1 HSC Q5a)

A particle is moving along the x axis starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of 5 m/s and an acceleration given by $\ddot{x} = 2x^3 + 2x$

(i) Show that $\dot{x} = x^2 + 1$

(ii) Hence find an expression for x in terms of t

$$v \frac{dv}{dx} = 2x^3 + 2x$$
$$\int_5^v v dv = \int_2^x (2x^3 + 2x) dx$$
$$\left[v^2 \right]_5^v = 2 \left[\frac{1}{2} x^4 + x^2 \right]_2^x$$

$$v^2 - 25 = x^4 + 2x^2 - 24$$

$$v^2 = x^4 + 2x^2 + 1$$

$$v^2 = (x^2 + 1)^2$$

$$\underline{v = x^2 + 1}$$

Note: $v > 0$, in order to satisfy initial conditions

$$\frac{dx}{dt} = x^2 + 1$$

$$\int_0^t dt = \int_2^x \frac{dx}{x^2 + 1}$$

$$t = \left[\tan^{-1} x \right]_2^x$$

$$t = \tan^{-1} x - \tan^{-1} 2$$

$$\tan^{-1} x = t + \tan^{-1} 2$$

$$x = \tan(t + \tan^{-1} 2)$$

$$\underline{x = \frac{\tan t + 2}{1 - 2 \tan t}}$$

Exercise 6A;
1 to 3abdf,
4ac, 5ac, 6bd,
7, 8, 9, 10,11,
13, 15, 17, 18,
19, 20, 21, 22