Normal Approximation to the Binomial

e.g. Estimate the probability that a school of 1200 students contains more than 150 left-handed students

Q: How would you solve such a problem?

A: One approach would be to take a large sample, say 50 students, and count the number of left handed students. From that information you could estimate the probability.

Say the sample contained 8 left handed students, we would estimate the probability of being left handed as;

$$P(\text{left handed}) = \frac{8}{50} = 0.16$$

Using binomial probability;

Let X = number of left handed students

$$X \sim \text{Bin}(1200, 0.16)$$

$$P(X > 150) = 1 - P(X \leq 150)$$

$$= 1 - \binom{1200}{0}(0.84)^{1200}(0.16)^{0} - \binom{1200}{1}(0.84)^{1199}(0.16)^{1} - \ldots$$

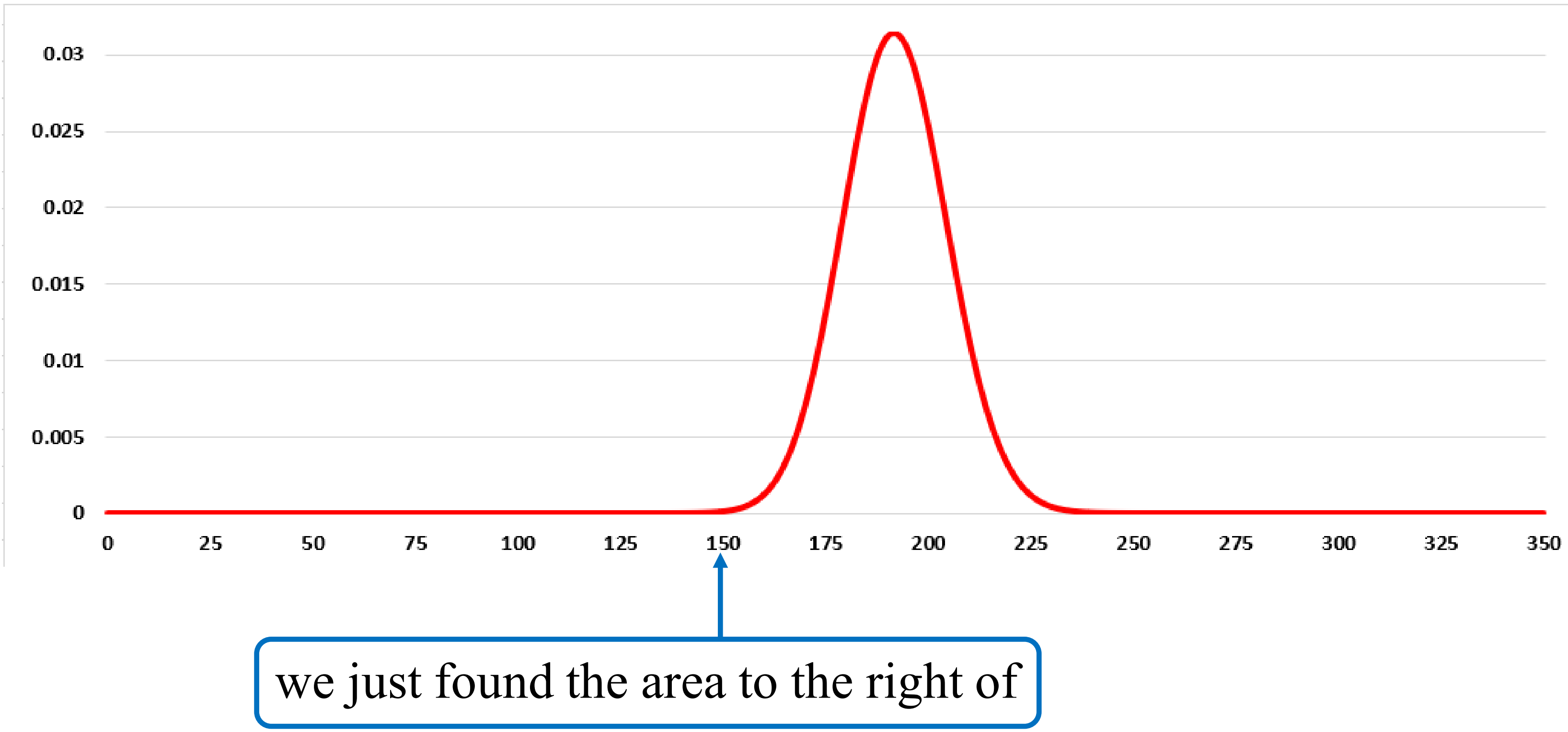
$$\ldots - \binom{1200}{149}(0.84)^{1051}(0.16)^{149} - \binom{1200}{150}(0.84)^{1050}(0.16)^{150}$$

even using the idea of complimentary events, it still involves 151 calculations

$$= 1 - 0.0003838$$

$$= 0.9996 \quad \text{(to 4 dp)}$$

The distribution's polygon would look like



- The distribution has a modal class somewhere in the middle of the range of values
- The distribution is symmetrical
- The frequency density tails off fairly rapidly as values move further away from the modal class

These are the features of a normal distribution

So what would have happened if we assumed our distribution was normal;

first we need to find the mean and variation

$X \sim \text{Bin}(1200, 0.16)$

$$\mu = np = (1200)(0.16) = 192$$

$$\sigma^2 = np(1-p) = (1200)(0.16)(0.84) = 161.28$$

$$X \sim N(192, 161.28) \Leftrightarrow Z \sim N(0,1) \text{ where } Z = \frac{X-\mu}{\sigma}$$

we need to standardise the data

$$\begin{aligned} P(X > 150) &= 1 - P(X \leq 150) \\ &= 1 - P\left(Z \leq \frac{150-192}{\sqrt{161.28}}\right) \\ &= 1 - \Phi(-3.31) \\ &= 1 - 0.0046654 \\ &= 0.9995 \text{ (to 4 dp)} \end{aligned}$$

% difference to binomial answer

$$\frac{0.999616326 - 0.99953346}{0.999616326} \times \frac{100}{1}$$

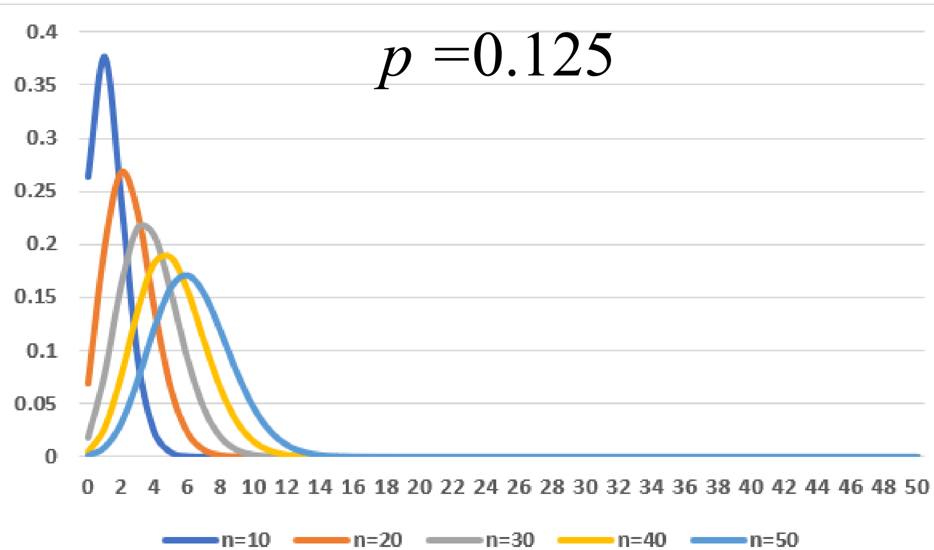
$$= 0.008\%$$

When is it okay to approximate a binomial distribution with a normal distribution?

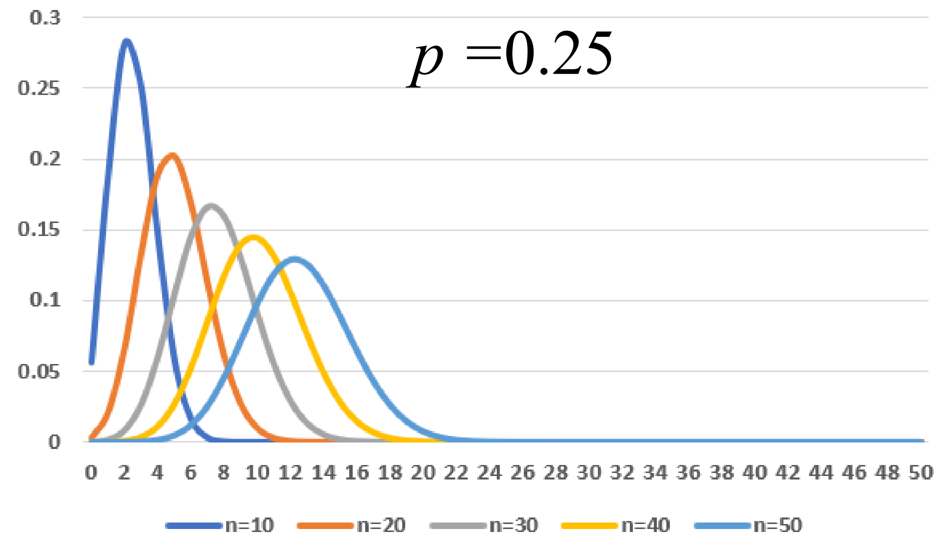
Keep in mind the three key features of a normal distribution

- The distribution has a modal class somewhere in the middle of the range of values
- The distribution is symmetrical
- The frequency density tails off fairly rapidly as values move further away from the modal class

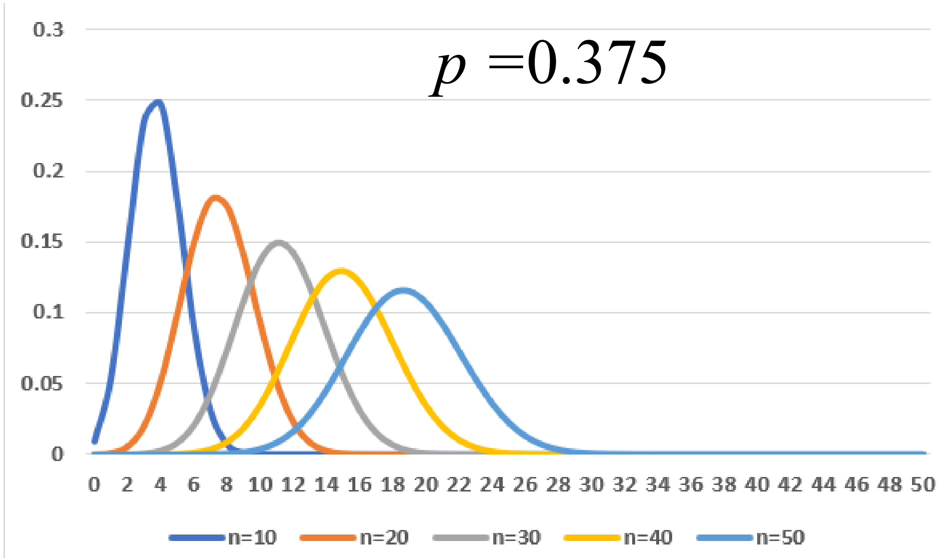
Let's take a look at some binomial distributions;



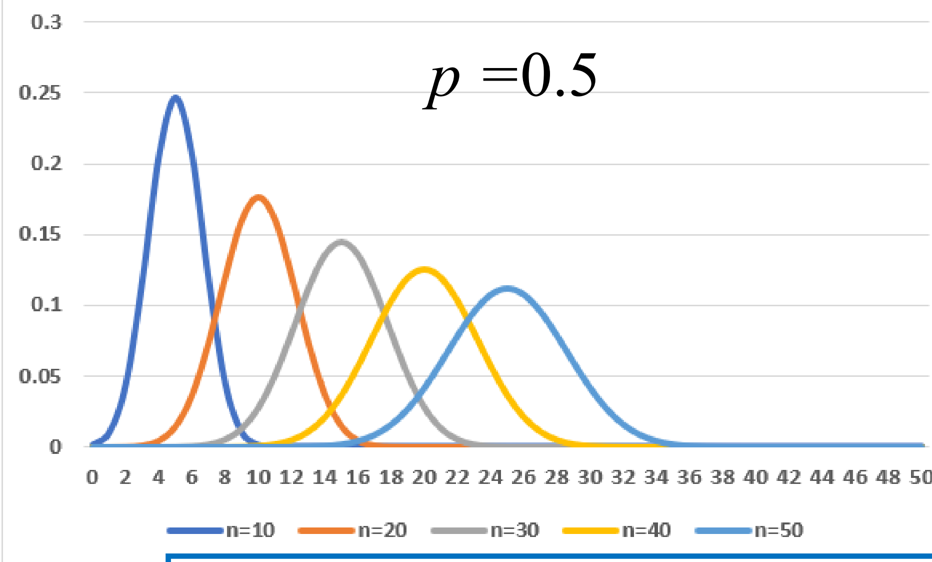
n	p	q	np	nq	☑ or ☒
10	0.125	0.875	1.25	8.75	☒
20	0.125	0.875	2.50	17.50	☒
30	0.125	0.875	3.75	26.25	☒
40	0.125	0.875	5.00	35.00	☑ ☒
50	0.125	0.875	6.25	43.75	☑



n	p	q	np	nq	☑ or ☒
10	0.25	0.75	2.5	7.5	☒
20	0.25	0.75	5.0	15.0	☑ ☒
30	0.25	0.75	7.5	22.5	☑
40	0.25	0.75	10.0	30.0	☑
50	0.25	0.75	12.5	37.5	☑



n	p	q	np	nq	☑ or ☒
10	0.375	0.625	3.75	6.25	☒
20	0.375	0.625	7.50	12.50	☑
30	0.375	0.625	11.25	18.75	☑
40	0.375	0.625	15.00	25.00	☑
50	0.375	0.625	18.75	31.25	☑



n	p	q	np	nq	☑ or ☒
10	0.5	0.5	5.00	5.00	☒ ☑
20	0.5	0.5	10.00	10.00	☑
30	0.5	0.5	15.00	15.00	☑
40	0.5	0.5	20.00	20.00	☑
50	0.5	0.5	25.00	25.00	☑

As a general rule; for a satisfactory approximation $np > 5$ and $nq > 5$

We are approximating the area under the polygon with bins 1 unit apart, so for the interval $a \leq x \leq b$, the area under the polygon is actually

$$\int_{a-0.5}^{b+0.5} f(x)dx$$

(this might be more obvious if you think of the histogram)

continuity correction for small samples

for small n; $P(a \leq X \leq b)$ use $P(a - 0.5 \leq X \leq b + 0.5)$

e.g. 2021 Extension 1 HSC Question 12b)

When a particular biased coin is tossed, the probability of obtaining a head is $\frac{3}{5}$. The coin is tossed 100 times.

Let X be the random variable representing the number of heads obtained. This random variable would have a binomial distribution.

(i) Find the expected value, $E(X)$.

$$
\begin{aligned}
E(X) &= np \\
&= 100 \times \frac{3}{5} \\
&= \underline{60}
\end{aligned}
$$

(ii) By finding the variance, Var(X), show that the standard deviation of X is approximately 5

$$
\begin{aligned}
\text{Var}(X) &= np(1-p) \\
&= 100 \times \frac{3}{5} \times \frac{2}{5} \\
&= 24
\end{aligned}
$$

$$
\begin{aligned}
\sigma &= \sqrt{\text{Var}(X)} \\
&= \sqrt{24} \\
&= 4.898979\ldots \underline{\approx 5}
\end{aligned}
$$

(iii) By using a normal approximation, find the approximate probability that X is between 55 and 65.

$$X \sim \mathrm{Bin}(100, 0.6)$$

$$X \sim \mathrm{N}(60, 24) \iff Z \sim \mathrm{N}(0, 1)$$

$$P(55 \le X \le 65) \approx P\left(\frac{55-60}{5} \le Z \le \frac{65-60}{5}\right)$$

$$= P(-1 \le Z \le 1)$$

$$= 0.68$$

2022 Extension 1 HSC Question 13e)

A chocolate factory sells 150 gram chocolate bars. There has been a complaint that the bars actually weigh less than 150 grams, so a team of inspectors was sent to the factory to check. They randomly selected 16 bars, weighed them and noted that 8 bars weighed less than 150 grams.

The factory manager claims 80% of the chocolate bars produced by the factory weigh 150 grams or more.

(i) The inspectors used the normal approximation to the binomial distribution to calculate the probability, P, of having at least 8 bars weighing less than 150 grams in a random sample of 16, assuming the factory manager's claim is correct.

Calculate the value of P.

$$X = \text{\# of bars weighing} < 150 \text{ grams}$$

$$X \sim \text{Bin}(16, 0.2)$$

$$\mu = 16 \times 0.2 = 3.2 \qquad \sigma^2 = 16 \times 0.2 \times 0.8 = 2.56$$

$$X \sim N(3.2, 2.56) \Leftrightarrow Z \sim N(0,1)$$

$$X \sim N(3.2, 2.56) \Leftrightarrow Z \sim N(0,1)$$

$$\begin{aligned} P(X \geq 8) &= 1 - P(X \leq 8) \\ &= 1 - P\left(Z \leq \frac{8 - 3.2}{\sqrt{2.56}}\right) \\ &= 1 - \Phi(3) \\ &= 1 - 0.9987 \\ &= \underline{0.0013} \end{aligned}$$

(ii) The factory manager disagrees with the method used by the inspectors as described in part (i).

Explain why the method used by the inspectors might not be valid.

$np = 3.2 < 5$

for a satisfactory approximation you would like $np \wedge nq > 5$

so it is unlikely that this would produce a satisfactory approximation

Exercise 17C;
1, 2adg, 3, 4, 5, 7, 9, 11, 12

https://www.desmos.com/calculator/qmt2h6n8cm
link for comparing approximations