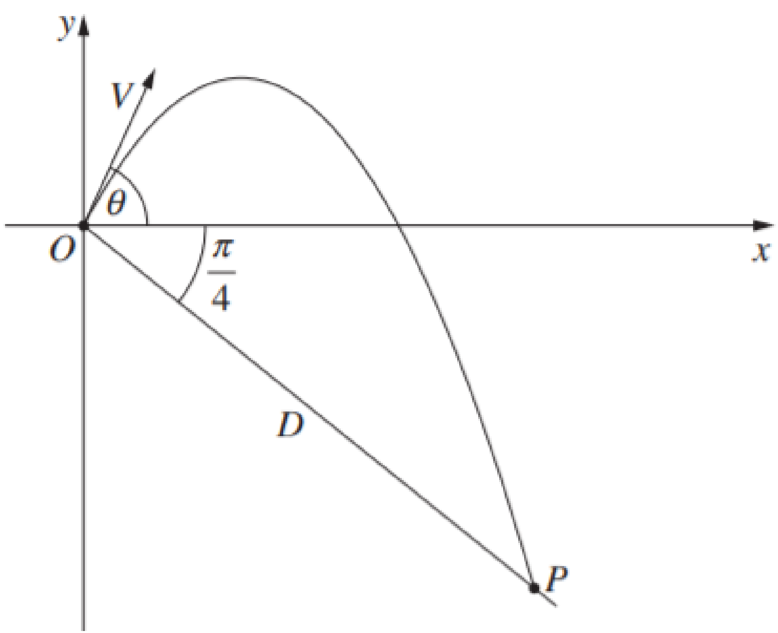
Further Projectile Motion

e.g. (i) 2014 Extension 1 HSC Q14 a)

The take-off point O on a ski jump is located at the top of a downslope. The angle between the downslope and the horizontal is $\frac{\pi}{4}$. A skier takes off from O with velocity V m/s at an angle θ to the horizontal, where $0 \leq \theta \leq \frac{\pi}{2}$

The skier lands on the downslope at some point P, a distance D metres from O.



The flight path of the skier is given by

$$x = Vt\cos\theta \quad , \quad y = \frac{1}{2}gt^2 + Vt\sin\theta$$

where t is the time in seconds after take-off.

a) Show that the Cartesian equation of the flight path of the skier is given by

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

$$x = Vt\cos\theta$$

$$t = \frac{x}{V\cos\theta}$$

$$y = -\frac{1}{2}gt^2 + Vt\sin\theta$$

$$= -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$$

$$= -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

b) Show that $D = 2\sqrt{2}\,\dfrac{V^2}{g}\cos\theta(\cos\theta + \sin\theta)$

$$\frac{y}{x} = \frac{-\frac{1}{2}gt^2 + Vt\sin\theta}{Vt\cos\theta}$$

$$\tan\left(-\frac{\pi}{4}\right) = \frac{-gt^2 + 2Vt\sin\theta}{2Vt\cos\theta}$$

$$-1 = \frac{-gt^2 + 2Vt\sin\theta}{2Vt\cos\theta}$$

$$2Vt\cos\theta = gt^2 - 2Vt\sin\theta$$

$$t^2 = \frac{2Vt}{g}(\sin\theta + \cos\theta)$$

$$t = \frac{2V}{g}(\sin\theta + \cos\theta) \qquad (\text{at } P,\ t > 0)$$

$$D = \sqrt{2}\,x$$

$$= \sqrt{2}\,Vt\cos\theta$$

$$= 2\sqrt{2}\,\frac{V^2}{g}\cos\theta(\sin\theta + \cos\theta)$$

c) Show that $\dfrac{dD}{d\theta} = 2\sqrt{2}\,\dfrac{V^2}{g}(\cos 2\theta - \sin 2\theta)$

$$D = 2\sqrt{2}\,\frac{V^2}{g}\cos\theta(\sin\theta + \cos\theta)$$

$$= 2\sqrt{2}\,\frac{V^2}{g}(\cos\theta\sin\theta + \cos^2\theta)$$

$$= 2\sqrt{2}\,\frac{V^2}{g}\left(\frac{1}{2}\sin 2\theta + \cos^2\theta\right)$$

$$\frac{dD}{d\theta} = 2\sqrt{2}\,\frac{V^2}{g}(\cos 2\theta - 2\cos\theta\sin\theta)$$

$$= 2\sqrt{2}\,\frac{V^2}{g}(\cos 2\theta - \sin 2\theta)$$

d) Show that D has a maximum value and find the value of θ for which this occurs.

$$D = 2\sqrt{2}\frac{V^2}{g}\cos\theta(\cos\theta + \sin\theta)$$

$$= \sqrt{2}\frac{V^2}{g}(\sin 2\theta + \cos 2\theta + 1)$$

$$= \frac{2V^2}{g}\cos\left(2\theta - \frac{\pi}{4}\right) + \sqrt{2}\frac{V^2}{g}$$

$\cos\left(2\theta - \frac{\pi}{4}\right)$ is a maximum when $2\theta - \frac{\pi}{4} = 0$

$$2\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8}$$

$\therefore D$ will be a maximum when $\theta = \frac{\pi}{8}$

(ii) A stone is thrown so that it will hit a bird at the top of a pole. However, at the instant the stone is thrown, the bird flies away in a horizontal straight line at a speed of 10 m/s.

The stone reaches a height double that of the pole and, in its descent, touches the bird.

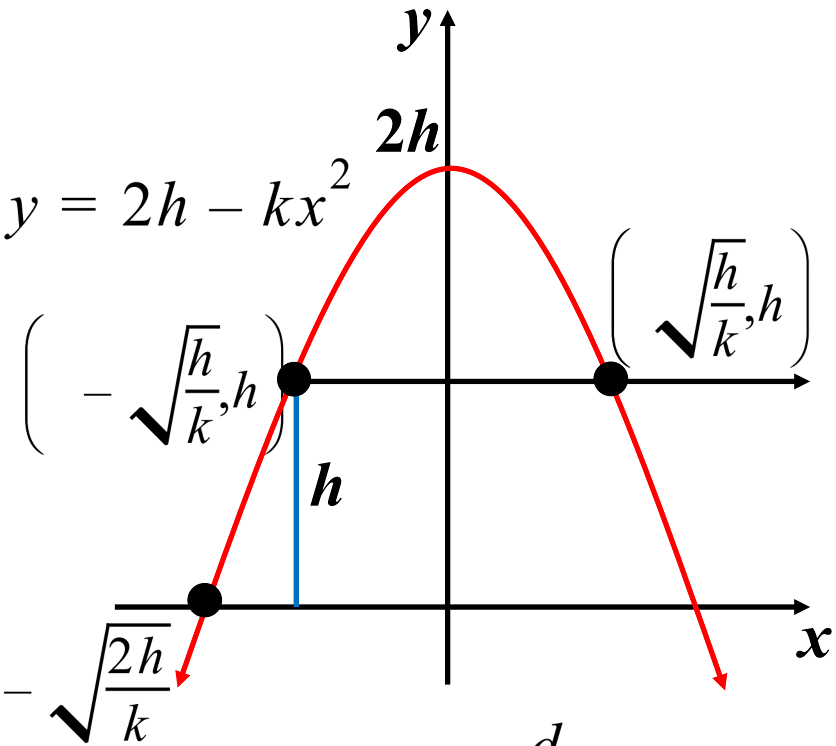
Find the horizontal component of the velocity of the stone.

Assuming their is no air resistance, the path of a projectile will be a parabola

sometimes your knowledge of the quadratic function, can assist in solving projectile motion questions

The greatest height will occur at the vertex of the parabola, so let the coordinates of the vertex be $(0,2h)$.

Thus the equation of the parabola will be $y = 2h - kx^2$



when $y = h$, $h = 2h - kx^2$

$$kx^2 = h$$

$$x^2 = \frac{h}{k}$$

$$x = \pm\sqrt{\frac{h}{k}}$$

so the bird flies a total of $2\sqrt{\frac{h}{k}}$ metres at $10\ ms^{-1}$

when $y = 0$, $0 = 2h - kx^2$

$$kx^2 = 2h$$

$$x^2 = \frac{2h}{k}$$

$$x = \pm\sqrt{\frac{2h}{k}}$$

$$\text{time} = \frac{d}{s}$$

$$= \frac{2\sqrt{\frac{h}{k}}}{10}$$

$$= \frac{1}{5}\sqrt{\frac{h}{k}}$$

the rock travels $\sqrt{\frac{2h}{k}} + \sqrt{\frac{h}{k}}$

$$= \sqrt{\frac{h}{k}}(\sqrt{2} + 1) \text{ metres horizontally}$$

$$s = \frac{d}{t}$$

$$= \frac{\sqrt{\frac{h}{k}}(\sqrt{2} + 1)}{\frac{1}{5}\sqrt{\frac{h}{k}}}$$

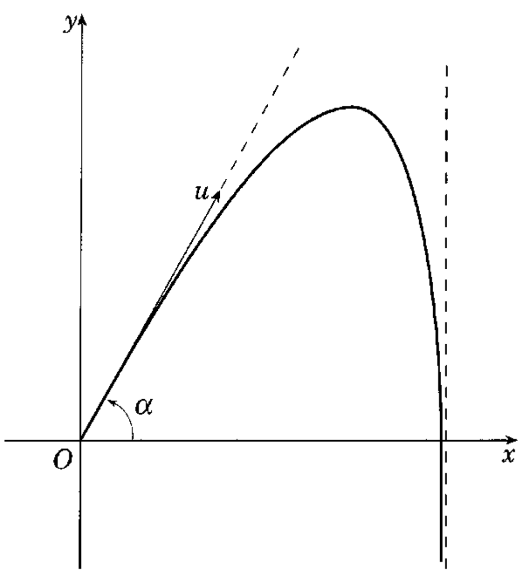
$$= 5(\sqrt{2} + 1)\ ms^{-1}$$

$\therefore$ horizontal velocity of the stone is $5(\sqrt{2} + 1)$ ms^{-1}

Projectile Motion & Resistance

2003 Extension 2 HSC Q5 b)

A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle of α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both directions.



The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g$$

where k is a constant and the acceleration due to gravity is g.

(You are NOT required to show these)

a) Derive the result $\dot{x} = ue^{-kt}\cos\alpha$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$

$$t = -\frac{1}{k}\int_{u\cos\alpha}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$

$$t = -\frac{1}{k}\left[\log \dot{x}\right]_{u\cos\alpha}^{\dot{x}}$$

$$t = -\frac{1}{k}\left[\log \dot{x} - \log(u\cos\alpha)\right]$$

$$t = -\frac{1}{k}\log\left(\frac{\dot{x}}{u\cos\alpha}\right)$$

$$-kt = \log\left(\frac{\dot{x}}{u\cos\alpha}\right)$$

$$\frac{\dot{x}}{u\cos\alpha} = e^{-kt}$$

$$\underline{\dot{x} = ue^{-kt}\cos\alpha}$$

b) Verify that $\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]$ satisfies the appropriate equation of motion and initial condition

$$\frac{d\dot{y}}{dt} = -k\dot{y} - g$$

$$t = -\int_{u\sin\alpha}^{\dot{y}} \frac{d\dot{y}}{k\dot{y} + g}$$

$$t = -\frac{1}{k}\left[\log(k\dot{y} + g)\right]_{u\sin\alpha}^{\dot{y}}$$

$$-kt = \log(k\dot{y} + g) - \log(ku\sin\alpha + g)$$

$$-kt = \log\left(\frac{k\dot{y} + g}{ku\sin\alpha + g}\right)$$

$$\frac{k\dot{y} + g}{ku\sin\alpha + g} = e^{-kt}$$

$$\underline{\dot{y} = \frac{1}{k}\left[(ku\sin\alpha + g)e^{-kt} - g\right]}$$

c) Find the value of t when the particle reaches its maximum height

Maximum height occurs when $\dot{y} = 0$

$$t = -\frac{1}{k}\left[\log(k\dot{y} + g)\right]_{u\sin\alpha}^{0}$$

$$t = -\frac{1}{k}\left[\log(g) - \log(ku\sin\alpha + g)\right]$$

$$t = \frac{1}{k}\log\left(\frac{ku\sin\alpha + g}{g}\right)$$

d) What is the limiting value of the horizontal displacement of the particle?

$$\dot{x} = ue^{-kt}\cos\alpha$$

$$\frac{dx}{dt} = ue^{-kt}\cos\alpha$$

$$x = \lim_{t\to\infty} u\cos\alpha\int_0^t e^{-kt}\,dt$$

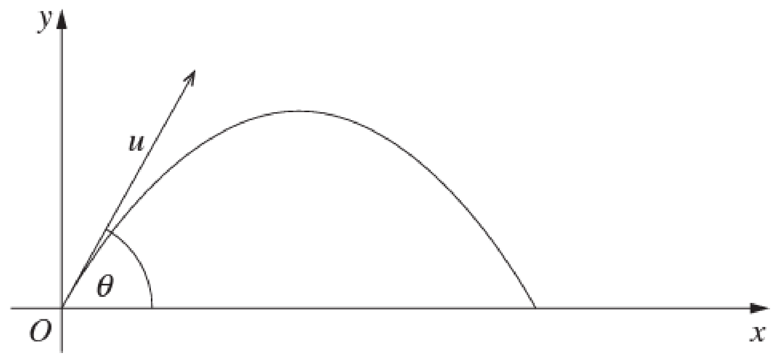
$$x = \lim_{t\to\infty} u\cos\alpha\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$x = \lim_{t\to\infty}\frac{u\cos\alpha}{k}\left(-e^{-kt} + 1\right)$$

$$x = \frac{u\cos\alpha}{k}$$

2021 Extension 2 HSC Q16 b)

A particle which is projected from the origin with initial speed u ms^{-1} at an angle of θ to the positive x-axis lands on the x-axis, as shown in the diagram. The particle is subject to an acceleration due to gravity of g ms^{-2}.



The position vector of the particle $\underset{\sim}{r}(t)$, where$t$ is the time in seconds after the particle is projected, is given by

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\frac{gt^2}{2} + ut\sin\theta \end{pmatrix} \quad \text{(Do NOT prove this)}$$

For some value(s) of θ there will be two times during the flight when the particle's position vector is perpendicular to its velocity vector.

Find the value(s) of θ for which this occurs, justifying that both times occur during the time of flight.

$$\underset{\sim}{r}(t) = \begin{pmatrix} ut\cos\theta \\ -\dfrac{gt^2}{2} + ut\sin\theta \end{pmatrix} \qquad \underset{\sim}{v}(t) = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$$

$$\underset{\sim}{r} \perp \underset{\sim}{v} \Rightarrow \underset{\sim}{r} \cdot \underset{\sim}{v} = 0$$

$$u^2 t\cos^2\theta + \frac{1}{2}g^2t^3 - \frac{1}{2}gut^2\sin\theta - gut^2\sin\theta + u^2 t\sin^2\theta = 0$$

$$u^2 t + \frac{1}{2}g^2t^3 - \frac{3}{2}gut^2\sin\theta = 0$$

however $t > 0$ as it is during the time of flight

$$g^2t^2 - 3gut\sin\theta + 2u^2 = 0$$

as there are two distinct times this happens, $\Delta > 0$

$$9g^2u^2\sin^2\theta - 8g^2u^2 > 0$$

$$9\sin^2\theta - 8 > 0$$

$$\sin^2\theta > \frac{8}{9}$$

$$\therefore \frac{8}{9} < \sin^2\theta < 1 \qquad (\theta \text{ is acute})$$

$$\sqrt{\frac{8}{9}} < \sin\theta < 90°$$

$$\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) < \theta < 90°$$

$$71° < \theta < 90° \qquad \text{(to the nearest degree)}$$

particle hits the ground when $y = 0$ i.e. $-\frac{1}{2}gt^2 + ut\sin\theta = 0$

$$-\frac{1}{2}gt + u\sin\theta = 0$$

$$t = \frac{2u\sin\theta}{g}$$

so both values of θ must occur when

$$0 < t \leq \frac{2u\sin\theta}{g}$$

$$g^2t^2 - 3gut\sin\theta + 2u^2 = 0$$

$$t = \frac{3gu\sin\theta \pm \sqrt{9g^2u^2\sin^2\theta - 8g^2u^2}}{2g^2}$$

$$= \frac{3u\sin\theta \pm u\sqrt{9\sin^2\theta - 8}}{2g}$$

now $3u\sin\theta - u\sqrt{9\sin^2\theta - 8} > 3u\sin\theta - u\sqrt{9\sin^2\theta}$

$$= 0$$

$\therefore$ both times are > 0

the larger value is

$$t = \frac{3u\sin\theta + u\sqrt{9\sin^2\theta - 8}}{2g}$$

$$< \frac{3u\sin\theta + u\sqrt{9\sin^2\theta - 8\sin^2\theta}}{2g} \qquad (\sin^2\theta \leq 1)$$

$$t < \frac{3u\sin\theta + u\sin\theta}{2g}$$

$$= \frac{2u\sin\theta}{g}$$

so both times occur within the time of flight

$$71° < \theta < 90°$$

Exercise 6F; 2, 4, 5, 8, 10, 11, 12, 14, 15, 16, 17, 19, 20

Exercise 6G; 2, 3, 5, 6, 7, 9, 12, 13, 16