Expanding Binomials

$$(1+x)^n = (1+x)(1+x)(1+x)(1+x)\cdots(1+x)$$

when expanding parentheses we choose a term from each set of parentheses and multiply them together.

$$x \times 1 \times x \times 1 \times 1 \times 1 \times \cdots \times x = x^3$$

Question: How many different ways could you end up with x^3 ?

OR How many different ways can you choose three x's from n sets of parentheses?

Answer: ${}^nC_3$

General Expansion of Binomials

${}^{n}C_k$ is the coefficient of x^k in $(1 + x)^n$

$$ {}^{n}C_k = \binom{n}{k} $$

$$ (1+x)^n = {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \ldots + {}^{n}C_n x^n $$

which extends to;

$$ (a+b)^n = {}^{n}C_0 a^n + {}^{n}C_1 a^{n-1} b + {}^{n}C_2 a^{n-2} b^2 + \ldots + {}^{n}C_{n-1} a b^{n-1} + {}^{n}C_n b^n $$

$$ e.g.(2+3x)^4 = {}^{4}C_0 2^4 + {}^{4}C_1 2^3 (3x) + {}^{4}C_2 2^2 (3x)^2 + {}^{4}C_3 2 (3x)^3 + {}^{4}C_4 (3x)^4 $$

$$ = 16 + 96x + 216x^2 + 216x^3 + 81x^4 $$

First Three Basic Pascal's Triangle Properties

$$(1)\quad {}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k} \quad \text{where } 1 \le k \le n-1$$

$$(1+x)^{n} = (1+x)(1+x)^{n-1}$$

$$= (1+x)\left({}^{n-1}C_{0} + {}^{n-1}C_{1}x + \ldots + {}^{n-1}C_{k-1}x^{k-1} + {}^{n-1}C_{k}x^{k} + \ldots + {}^{n-1}C_{n-1}x^{n-1}\right)$$

looking at coefficients of x^{k}

$$LHS = {}^{n}C_{k} \qquad RHS = (1)\left({}^{n-1}C_{k-1}\right) + (1)\left({}^{n-1}C_{k}\right)$$

$$= {}^{n-1}C_{k-1} + {}^{n-1}C_{k} \qquad \therefore {}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$$

$$(2)\quad {}^{n}C_{k} = {}^{n}C_{n-k} \quad \text{where } 1 \le k \le n-1$$

"Pascal's triangle is symmetrical"

$$(3)\quad {}^{n}C_{0} = {}^{n}C_{n} = 1$$

Expanding Perfect Parentheses

$$(a+b)^2 = a^2 + 2ab + b^2$$

A different way of thinking about it

$$(a+b)^2$$

$$= \underline{\ 1\ }\left(a^2+b^2\right) + \underline{\ 2!\ }\,ab$$

$$= \left(a^2+b^2\right) + 2ab$$

1. What are all the different ways of writing two pronumerals using ***a*** and ***b***?
2. How many ways can you arrange **two *a*'s** or **two *b*'s**
3. How many ways can you arrange **one *a* and one *b***

$$(a+b+c+\ldots+n)^2$$

$$= \left(a^2+b^2+c^2+\ldots+n^2\right) + 2\left(ab+ac+an+bc+bn+\ldots+cn\right)$$

$(a+b)^3$

1. What are all the different ways of writing three pronumerals using ***a*** and ***b***?

$$= \underline{\ 1\ }\left(a^3+b^3\right)+\underline{\frac{3!}{2!}}\left(ab^2+a^2b\right)$$

2. How many ways can you arrange **three *a*’s** or **three *b*’s**
3. How many ways can you arrange **two *a*’s and one *b*** or **two *b*’s and one *a***

$$=\left(a^3+b^3\right)+3\left(a^2b+ab^2\right)$$

$(a+b+c)^3$

$$= \underline{\ 1\ }\left(a^3+b^3+c^3\right)+\underline{\frac{3!}{2!}}\left(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2\right)+\underline{\ 3!\ }\,abc$$

$$=\left(a^3+b^3+c^3\right)+3\left(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2\right)+6abc$$

$(a+b)^4$

$$= \underline{\ 1\ }\left(a^4+b^4\right)+\underline{\frac{4!}{3!}}\left(ab^3+a^3b\right)+\underline{\frac{4!}{2!2!}}\,a^2b^2$$

$$=\left(a^4+b^4\right)+4\left(ab^3+a^3b\right)+6a^2b^2$$

eg Expand $(a + b + c)^4$

$$(a + b + c)^4 = (a^4 + b^4 + c^4) + \frac{4!}{3!}(ab^3 + ac^3 + a^3b + bc^3 + a^3c + b^3c) + \frac{4!}{2!}(abc^2 + ab^2c + a^2bc) + \frac{4!}{2!2!}(a^2b^2 + a^2c^2 + b^2c^2)$$

$$= a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4ac^3 + b^4 + 4b^3c + 6b^2c^2 + 4bc^3 + c^4$$

Exercise 15B; 2ace, 3, 4ac, 5, 6bd, 8ac, 9, 11, 12, 15