Further Growth & Decay

In order to take into account other conditions *(e.g. the temperature of an object will never be lower than the temperature of its surroundings)*, we need to change the equation.

$$\frac{dP}{dt} = k(P - N)$$

the solution to the equation is;

$$P = N \qquad \textit{(trivial case)}$$

$$OR$$

$$P = N + Ae^{kt}$$

P = population at time t

k = growth(or decay) constant

$N + A$ = initial population

t = time

e.g.(1993)

Let T be the temperature inside a room at time t and let A be the constant outside air temperature.
Newton's law of cooling states that the rate of change of the temperature is proportional to $(T - A)$.

a) Show that $T = A + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

$$T = A + Ce^{kt}$$

$$\frac{dT}{dt} = kCe^{kt}$$

$$\text{but } T - A = Ce^{kt}$$

$$\frac{dT}{dt} = k\left(T - A\right)$$

$$\therefore T = A + Ce^{kt} \text{ satisfies } \frac{dT}{dt} = k\left(T - A\right)$$

b) The outside air temperature is $5^\circ C$ and a heating system breakdown causes the inside room temperature to drop from $20^\circ C$ to $17^\circ C$ in half an hour.

After how many hours is the inside room temperature equal to $10^\circ C$?

$$T = 5 + Ce^{kt}$$

when $t = 0, T = 20$

$$\therefore C = 15$$

$$T = 5 + 15e^{kt}$$

when $t = 0.5, T = 17$

i.e. $17 = 5 + 15e^{0.5k}$

$$15e^{0.5k} = 12$$

$$e^{0.5k} = \frac{12}{15}$$

$$0.5k = \log\left(\frac{12}{15}\right)$$

$$k = 2\log\left(\frac{12}{15}\right)$$

$$\text{when } T = 10, \quad 10 = 5 + 15e^{kt}$$

$$15e^{kt} = 5$$

$$e^{kt} = \frac{1}{3}$$

$$kt = \log\frac{1}{3}$$

$$t = \frac{1}{k}\log\frac{1}{3}$$

$$t = 2.46167$$

$\therefore$ After $2\frac{1}{2}$ hours the temperature has dropped to $10^\circ C$

Exercise 16C; 2, 4, 5, 9, 10, 11, 13