Relationships Between Binomial Coefficients

Binomial Theorem $$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_k x^k + \dots + {}^nC_n x^n$$

e.g. (i) Find the values of;

a) $\sum_{k=1}^{n} {}^nC_k$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

let $x = 1$; $$(1+1)^n = \sum_{k=0}^{n} {}^nC_k 1^k$$

$$2^n = \sum_{k=0}^{n} {}^nC_k$$

$$2^n = {}^nC_0 + \sum_{k=1}^{n} {}^nC_k$$

$$\sum_{k=1}^{n} {}^nC_k = 2^n - {}^nC_0$$

$$\sum_{k=1}^{n} {}^nC_k = 2^n - 1$$

b) ${}^{n}C_1 + {}^{n}C_3 + {}^{n}C_5 + {}^{n}C_7 + \ldots\ldots$

$$(1+x)^n = \sum_{k=0}^{n} {}^{n}C_k x^k$$

$$= {}^{n}C_0 + {}^{n}C_1 x + {}^{n}C_2 x^2 + {}^{n}C_3 x^3 + {}^{n}C_4 x^4 + {}^{n}C_5 x^5 + \ldots\ldots$$

let $x = 1$; $(1+1)^n = {}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + {}^{n}C_4 + {}^{n}C_5 + \ldots\ldots$

$$2^n = {}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + {}^{n}C_4 + {}^{n}C_5 + \ldots\ldots \qquad (1)$$

let $x = -1$; $(1-1)^n = {}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - {}^{n}C_3 + {}^{n}C_4 - {}^{n}C_5 + \ldots\ldots$

$$0 = {}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - {}^{n}C_3 + {}^{n}C_4 - {}^{n}C_5 + \ldots\ldots \qquad (2)$$

subtract (2) from (1)

$$2^n = 2\,{}^{n}C_1 + 2\,{}^{n}C_3 + 2\,{}^{n}C_5 + \ldots\ldots$$

$$\underline{2^{n-1} = {}^{n}C_1 + {}^{n}C_3 + {}^{n}C_5 + \ldots\ldots}$$

c) $\sum_{k=1}^{n} k\,{}^{n}C_{k}$

$$(1+x)^{n} = \sum_{k=0}^{n} {}^{n}C_{k}x^{k}$$

Differentiate both sides

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_{k}x^{k-1}$$

let $x = 1$;

$$n(1+1)^{n-1} = \sum_{k=0}^{n} k\,{}^{n}C_{k}$$

$$n(2)^{n-1} = (0)\,{}^{n}C_{0} + \sum_{k=1}^{n} k\,{}^{n}C_{k}$$

$$\sum_{k=1}^{n} k\,{}^{n}C_{k} = n(2)^{n-1}$$

(ii) By equating the coefficients of x^n on both sides of the identity;

$$(1+x)^n(1+x)^n \equiv (1+x)^{2n}$$

show that;

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \frac{(2n)!}{(n!)^2}$$

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$$

$$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n$$

coefficient of x^n in $(1+x)^n(1+x)^n$

$$\left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n\right)$$

$$\times\left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_{n-2}x^{n-2} + {}^nC_{n-1}x^{n-1} + {}^nC_n x^n\right)$$

$$= \binom{n}{0}\binom{n}{n}x^n + \binom{n}{1}x\binom{n}{n-1}x^{n-1} + \binom{n}{2}x^2\binom{n}{n-2}x^{n-2}$$

$$+ \ldots + \binom{n}{n-2}x^{n-2}\binom{n}{2}x^2 + \binom{n}{n-1}x^{n-1}\binom{n}{1}x + \binom{n}{n}x^n\binom{n}{0}$$

$$\text{coefficient of } x^n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots + \binom{n}{n}\binom{n}{0}$$

But $\binom{n}{k} = \binom{n}{n-k}$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2$$

coefficient of x^n in $(1+x)^{2n}$

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \ldots + \binom{2n}{n}x^n + \ldots + \binom{2n}{2n}x^{2n}$$

$$\text{coefficient of } x^n = \binom{2n}{n}$$

Now $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$

$$\therefore \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

$$= \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)!}{(n!)^2}$$

(iii) 2020 Extension 1 HSC Question 14a)

a) Use the identity $(1 + x)^{2n} = (1 + x)^n(1 + x)^n$

to show that $\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2$

where n is a positive integer

this is essentially the same as the last example

b) A club has $2n$ members, with n women and n men.

A group consisting of an even number $(0, 2, 4, \ldots, 2n)$ of members is chosen, with the number of men equal to the number of women.

Show, giving reasons, that the number of ways to do this is $\binom{2n}{n}$

ways of choosing k men from a group of n men $= \binom{n}{k}$

Similarly;

ways of choosing k women from a group of n women $= \binom{n}{k}$

ways of choosing k men from a group of n men and choosing k women from a group of n women $= \binom{n}{k} \times \binom{n}{k}$

$$= \binom{n}{k}^2$$

Total ways of choosing the same number from both groups $= \sum_{k=0}^{n} \binom{n}{k}^2$

$$= \binom{2n}{n}$$

c) From the group chosen in part b), one of the men and one of the women are selected as leaders.

Show, giving reasons, the number of ways to choose the even number of people and then the leaders is

$$1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \ldots + n^2\binom{n}{n}^2$$

ways of choosing the leaders $= \binom{n}{k}^2 \times k \times k$

$$\text{Total ways} = \sum_{k=1}^{n} k^2 \binom{n}{k}^2$$ (note: $k = 0$ not possible as you must have someone to choose)

$$= 1^2\binom{n}{1}^2 + 2^2\binom{n}{2}^2 + \ldots n^2\binom{n}{n}^2$$

d) The process is now reversed so that the leaders, one man and one woman, are chosen first. The rest of the group is then selected, still made up of an equal number of women and men.

By considering this reversed process and using part b), find a simple expression for the sum in part c).

By choosing the leaders first you will always have n possibilities for the male leader and n possibilities for the female leader

To complete the group we must now choose k - 1 men from the remaining $n - 1$ men

i.e. $\binom{n-1}{k-1}$ ways

and the same for the women

$$\text{Total ways} = \sum_{k=1}^{n} n^2 \binom{n-1}{k-1}^2$$

$$= n^2 \binom{n-1}{0}^2 + n^2 \binom{n-1}{1}^2 + \ldots + n^2 \binom{n-1}{n-1}^2$$

$$= n^2 \left[\binom{n-1}{0}^2 + \binom{n-1}{1}^2 + \ldots + \binom{n-1}{n-1}^2 \right]$$

$$= n^2 \binom{2(n-1)}{n-1}$$

$$= n^2 \binom{2n-2}{n-1}$$

Exercise 15D;
1, 2, 4, 5, 6, 7, 9, 11, 12