Dot (Scalar) Product

Let $\underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

then;

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 x_2 + y_1 y_2$$

NOTE: the result is a scalar

e.g. $\left(\underset{\sim}{i} - 7\underset{\sim}{j}\right) \cdot \left(6\underset{\sim}{i} + 4\underset{\sim}{j}\right) = 1 \times 6 + (-7) \times 4$

$= \underline{-22}$

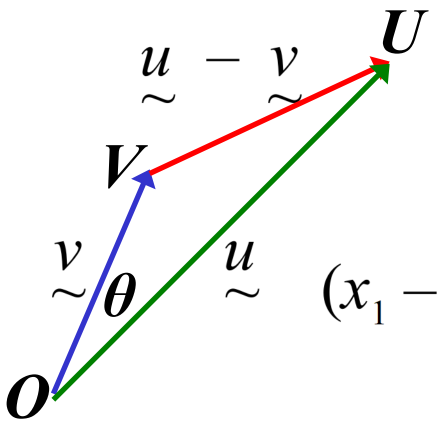
$$\lambda \underset{\sim}{u} \cdot \underset{\sim}{v} = \lambda(\underset{\sim}{u} \cdot \underset{\sim}{v})$$

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = x_1^{\,2} + y_1^{\,2}$$
$$= |\underset{\sim}{u}|^2$$

$$\underset{\sim}{a} \cdot (\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{a} \cdot \underset{\sim}{u} + \underset{\sim}{a} \cdot \underset{\sim}{v}$$

$$(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{v}$$
$$= |\underset{\sim}{u}|^2 - |\underset{\sim}{v}|^2$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$$



By the cosine rule;

$$|\underset{\sim}{u} - \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = {x_1}^2 + {x_2}^2 + {y_1}^2 + {y_2}^2 - 2|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

$$-2x_1\, x_2 - 2y_1\, y_2 = -2|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

$$2(\underset{\sim}{u} \cdot \underset{\sim}{v}) = 2|\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

Let $\underset{\sim}{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

then;

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 x_2 + y_1 y_2$$

$$= |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$$

NOTE: θ is acute or obtuse

e.g. Find, to the nearest degree, the angle between the two vectors

$$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} \text{ and } \underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\cos\theta = \frac{3 \times 4 + (-2) \times 1}{\sqrt{13} \times \sqrt{17}}$$

$$= \frac{10}{\sqrt{221}}$$

$$\underline{\theta = 48^\circ} \quad \text{(to nearest degree)}$$

Consequences of the Dot Product

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0 \iff \underset{\sim}{u} \perp \underset{\sim}{v}$$

$$\therefore \underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{i} = 0$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \pm|\underset{\sim}{u}||\underset{\sim}{v}| \iff \underset{\sim}{u} \parallel \underset{\sim}{v}$$

$|\underset{\sim}{u}||\underset{\sim}{v}| > 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have the same direction

$|\underset{\sim}{u}||\underset{\sim}{v}| < 0 \Rightarrow \underset{\sim}{u}$ and $\underset{\sim}{v}$ have opposite directions

$$\therefore \underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = 1$$

Exercise 8C; 1ac, 2a, 3a, 4b, 5ac, 6, 8, 9b, 10,11 abc (i, iv, vi), 12, 13, 15, 17, 20, 21

e.g. **Extension 1 2022 HSC Q11d)**

The vectors $\underset{\sim}{u} = \begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix}$

are perpendicular

What are the possible values of a?

$$\begin{pmatrix} a \\ 2 \end{pmatrix} \cdot \begin{pmatrix} a-7 \\ 4a-1 \end{pmatrix} = 0$$

$$(a^2 - 7a) + (8a - 2) = 0$$

$$a^2 + a - 2 = 0$$

$$(a+2)(a-1) = 0$$

$$\underline{a = -2 \quad \text{or} \quad a = 1}$$